Seminar on **Seiberg-Witten Theory**

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Section 1
4-dimensional manifolds

We are interested in classifying 4-manifolds. Without further constraints on the manifold, this is impossible, given the following theorems:

THM 1.1 There is *no* algorithm which, given a presentation of a group, whether or not it's trivial.

THM 1.2 Given a finitely presented group G, there exists a smooth 4-manifold with fundamental group G.

Proof. Suppose $G = \langle g_1, \cdots, g_n : r_1, \cdots, r_n \rangle$. By Seifert-van Kampen theorem, given 4-manifolds M and N, $\pi_1(M \# N) = \pi_1(M) * \pi_1(N)$. Consider $\#_n \mathbb{S}^1 \times \mathbb{S}^3$; it has fundamental group $N = \mathbb{Z}^{*n}$. Every relation r_i then represents a loop γ_i in N. To quotient them out, first find a tubular neighbourhood U_i of γ_i for every i. It is homeomorphic to $\mathbb{S}^1 \times \mathbb{D}^3$, with boundary $\mathbb{S}^1 \times \mathbb{S}^2$. Remove U_i from N and in its place we glue back $\mathbb{D}^2 \times \mathbb{S}^2$, whose boundary is also $\mathbb{S}^1 \times \mathbb{S}^2$. The latter is contractible, so again by Seifert-van Kampen theorem the resulting manifold has fundamental group G. □

This argument does not work for dimension 3, for the following reason:

THM 1.3 There is no simply-connected 3-manifold with boundary $\mathbb{S}^1 \times \mathbb{S}^1$

Proof. All compact odd-dimensional manifolds satisfy $\chi(M) = \frac{1}{2}\chi(\partial M)$. Since $\partial M = \mathbb{S}^1 \times \mathbb{S}^1$, $\chi(M) = \frac{1}{2}\chi(\mathbb{S}^1 \times \mathbb{S}^1) = 0$. An analysis on homology groups shows a contradiction. □

Therefore, we put our focus on a smaller subset of 4-manifold, simply-connected closed 4-manifolds M. Some observations are immediate. First, since $\pi_1(M) = 0$, we know $H_1(M) = H^1(M) = 0$, and by Poincaré duality $H_3(M) = 0$ and $H_2(M)$ is torsion-free. Second, again by Poincaré duality, $H_2(M) \cong H^2(M)$, which is the only interesting homology group.

DEF 1.4 intersection form

The Poincaré pairing map $Q: H_2 \times H_2 \to \mathbb{Z}$ is called the **intersection form**. It always has determinant ± 1, and is a perfect symmetric bilinear form.

For example,

- $X = \mathbb{S}^4$, $Q = 0$;
- $X = \mathbb{C}\mathrm{P}^2$, $Q = (1)$;
- $X = \overline{\mathbb{C}\mathrm{P}^2}$ ($\mathbb{C}\mathrm{P}^2$ with reversed orientation), $Q = (-1)$;
- $X = \mathbb{S}^2 \times \mathbb{S}^2$, $Q = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$;
- $X = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$, $Q = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$;
- Both $(\mathbb{S}^2 \times \mathbb{S}^2) \# \overline{\mathbb{C}\mathrm{P}^2}$ and $\mathbb{C}\mathrm{P}^2 \# 2\overline{\mathbb{C}\mathrm{P}^2}$ has Q congruent (over $\mathbb{Z}$) to $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$.

Q is called *intersection* form for the following reason.

THM 1.5

Suppose X is a smooth 4-manifold. Then every homology class in $H_2(X)$ is represented by some smoothly embedded 2-surface, and the Poincaré pairing of two homology classes corresponds to the intersection index of two transverse such surfaces.

Sketch of proof. There exists an isomorphism from the set of complex line bundles over X to $H^2(X;\mathbb{Z})$, taking a complex line bundle to its first Chern class. This is because complex m-dimensional vector bundles L on X correspond to homotopy classes of maps from X to the classification space $\mathrm{Gr}_m(\mathbb{C}^\infty)$, and when $m = 1$, $\mathrm{Gr}_m(\mathbb{C}^\infty) = \mathbb{C}\mathrm{P}^1 = K(\mathbb{Z},2)$. Therefore, complex line bundles corresponds to $\langle X, K(\mathbb{Z},2)\rangle \cong H^2(X;\mathbb{Z})$.

Gr denotes the Grassmanian

Therefore, for a homology class $\alpha \in H^2(X;\mathbb{Z})$, consider a complex line bundle $L \to X$ representing α. Let s be a generic section $X \to L$ transverse to the zero section. In real dimensions, $\dim X = 4$, $\dim L = 6$, so $s^{-1}(0)$ has dimension 2. This submanifold corresponds to α.

Alternatively, $H^2(X;\mathbb{Z}) \cong [X, \mathbb{C}\mathrm{P}^\infty] \cong \left[X, \mathbb{C}\mathrm{P}^2\right]$, and the pullback of $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2$ is the desired submanifold. □

The question is then how much information does the intersection form carry. In other words, whether it determines homotopy / homeomorphism / diffeomorphism type. On homotopy type the following theorem gives the answer:

THM 1.6 Whitehead's theorem

Two closed simply-connected 4-manifolds are homotopy equivalent if and only if their intersection forms are congruent over $\mathbb{Z}$.

On homeomorphism type, the answer depends on parity and smoothness. A bilinear form $A: \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ is called **even** if $A(a,a) \equiv 0 \pmod 2$ for all a, and **odd** otherwise. Parity is an invariant under congruence.

THM 1.7 (Freedman, 1982)

1. For any unimodular symmetric bilinear form Q, there exists a topological simply-connected closed 4-manifold X with intersection form Q;
2. if Q is even, then X is unique up to homeomorphism; if Q is not even, then there exists exactly 2 such X, and at most one of them admits a smooth structure.

Therefore, simply-connected closed *smooth* 4-manifolds are determined uniquely by their intersection form up to *homeomorphism*.

Diffeomorphism type is much more complicated and is core to the study of 4-manifolds. In dimension 3, every topological manifold admits a unique smooth structure up to diffeomorphism (Moise). This is far from true in dimension 4; for example, $\mathbb{R}^n$ has a unique smooth structure *except for* $n = 4$, where one gets infinitely many non-diffeomorphic smooth structures on $\mathbb{R}^4$ (Donaldson *et al.*). If X is a closed 4-manifold, it admits at most countably many smooth structures, and there *are* closed 4-manifolds which admits countably infinitely many smooth structures, for example $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$. We don't even know whether there are 4-manifolds with only finitely many smooth structures. For another example, for $n = 1, 2, 3, 5, 6$, $\mathbb{S}^n$ has a unique smooth structure; for $n = 7$ we have 28 smooth structures, given by Milnor's exotic spheres; for $n = 4$ this is still open.

The next question is naturally the classification of unimodular symmetric bilinear forms over $\mathbb{Z}$. Over $\mathbb{R}$, the classification of perfect symmetric bilinear forms is rather simple: they are classified by their signature. Over $\mathbb{Z}$ things are more complicated and more invariants are involved.

THM 1.8

1. If Q is indefinite and odd, then $Q = m \cdot (1) \oplus n \cdot (-1)$ $(m, n > 0)$;
2. if Q is indefinite and even, then $Q = m \cdot \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \oplus n \cdot E_8$ $(m > 0)$, where

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & & & & & \\ 0 & 2 & 0 & -1 & & & & \\ -1 & 0 & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

For Q definite, things are much more definite and is not determined by parity, rank and signature alone; for example, $9 \cdot (1)$ and $E_8 \oplus (1)$ are both odd, positive definite and has rank 9, but they are not equivalent over $\mathbb{Z}$.

When can Q become the intersection form of a *smooth* 4-manifold?

THM 1.9 (Roknlin, 1952)

If X is smooth and simply-connected and Q is even, then 16 divides the signature of Q.

Using this we can construct topological manifolds without smooth structures, for example, the topological manifold with intersection form E_8. But what about $E_8 \oplus E_8$? It has signature 16 and is even, but still it doesn't correspond to any smooth 4-manifold.

THM 1.10 Donaldson diagonalization theorem

If X is a closed simply-connected 4-manifold and if Q is definite, then it is diagonalizable over $\mathbb{Z}$. Therefore, $Q = \pm r \cdot (1)$.

Therefore, although algebraically we failed at classifying definite forms, the geometric constraint solves the problem. Putting everything together,

COR 1.11

The homeomorphism type of a simply-connected closed smooth 4-manifold X is determined uniquely by the signature $\sigma(Q)$, parity of Q, and $\chi(X)$. Equivalently, it's determined uniquely by the positive and negative index of Q and parity of Q.