Algebraic Topology I

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Section 1
Homology Theory

1.1 Simplicial Homology

An n**-simplex** is the smallest conves set containing $(n+1)$ points, $v_1, \cdots, v_n$ in general position, and is denoted by $[v_0, \cdots, v_n]$.

The **standard** n**-simplex** is the set

$$\Delta^n = \left\{(t_0, \cdots, t_n) \in \mathbb{R}^n : \sum t_i = 1, t_i \geqslant 0\right\}.$$

The order of the vertices is important. By convention, the edges of a simplex is oriented so that v_i points to v_j if $i < j$.

It is easily seen that all n-simplices are (linearly) homeomorphic to the standard n-simplex, and such a homeomorphism is determined by the vertices. As such, we often simply call any n-simplex Δ^n.

The $(n-1)$-simplex $\left[v_0, \cdots, \hat{v}_j, \cdots, v_n\right]$ is a **face** of the n-simplex. The union of all faces form the **boundary**, and is denoted $\partial\Delta^n$. The **open simplex** is $\mathring{\Delta}^n = \Delta^n \setminus \partial\Delta^n$. (The open simplex for the 0-simplex is defined to be the point itself.)

A Δ-complex structure on a topological space X is a collection of maps $\{\sigma_\alpha : \Delta^n \to X\}_{\alpha \in I}$ such that:

1. $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and every point of X is covered by *exactly one* image of such maps.
2. Each restriction of σ_α to a face of Δ^n is also in the set.
3. A subset $A \subseteq X$ is open, if and only if for any α, $\sigma_\alpha^{-1}(A)$ is open (in Δ^n).

For a space X with a Δ-complex structure, we define $C_n^\Delta(X)$ to be the free abelian group generated by its n-simplices. Elements in $C_n^\Delta(X)$ has the form $\sum_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^n \to X$ and $n_\alpha \in \mathbb{Z}$. Elements in $C_n^\Delta(X)$ are called the n **-chains**.

We define the **boundary map** $\partial_n : C_n^\Delta(X) \to C_{n-1}^\Delta(X)$ as follows. For an n-simplex $[v_0, \cdots, v_n]$, $\partial_n([v_0, \cdots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n]$, i.e. maps a simplex to its (signed) boundary. It's easy to see that $\partial_n \circ \partial_{n+1} = 0$.

DEF 1.1 simplicial homology

The groups $C_\bullet^\Delta$ and maps $\partial_\bullet$ together makes a **chain complex** (i.e. $\partial \circ \partial = 0$), called the **simplicial chain complex** of X. The homology groups of a chain complex is defined as $\ker \partial_n / \operatorname{im} \partial_{n+1}$, which are called the **simplicial homology groups** of X:

$$H_n^\Delta(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

1.2 Singular Homology

1.2.1 Definition and Properties

DEF 1.2 singular homology

Define $C_n(X)$ to be the free abelian group generated by the set of continuous maps $\sigma : \Delta^n \to X$. Elements of $C_n(X)$ are called ***n*-chains**. Define the boundary map as before. Again, the groups $C_\bullet$ and $\partial_\bullet$ makes a chain complex, called the **singular chain complex** of X. Its homology groups are called the **singular homology groups** of X, $H_n(X)$.

Singular homology groups are obviously topological invariants. It's also easier to study its properties.

PROP 1.3

1. The homology groups of X is the direct sum of the homology groups of its path-connected components.
2. If X is non-empty and path-connected, then $H_0(X) \cong \mathbb{Z}$.
3. If X is a single point, then $H_n(X) = 0$ for all $n > 0$.

We can attach an additional $\mathbb{Z}$ to the end of the chain complex:

$$\cdots \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0$$

to eliminate the $\mathbb{Z}$ at H_0. The homology groups of this extended chain complex are called the **reduced homology groups** $\tilde{H}_n$. For $n = 0$, $\tilde{H}_0 \oplus \mathbb{Z} \cong H_0$; for $n > 1$, $\tilde{H}_n \cong H_n$.

THM 1.4 functoriality

The homology groups, $H_n(\cdot)$, are functors from Top (category of topological spaces and continuous maps) to AbGp (category of abelian groups and group homomorphisms).

To prove this, we need some algebra. A **chain map** between chain complexes $C_\bullet \xrightarrow{f} C'_\bullet$ is such a map that $\partial \circ f = f \circ \partial'$; i.e. the following diagram commutes.

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \end{tikzcd}$$

It's easy to deduce that a chain map f can induce a homomorphism between the homology groups: $f_* : H_n(C) \to H_n(C')$. A continuous map $f : X \to Y$ induces a chain map between the singular chain complexes of X and Y (just by composing), so it in turn induces a homomorphism between their singular homology groups.

THM 1.5 homotopy invariance

If two maps $f, g : X \to Y$ are homotopic, then they induces the same homomorphisms between the homology groups: $f_* = g_*$.

The algebraic counterpart is as follows. A **chain homotopy** between two chain maps $f, g : (C_\bullet, \partial_\bullet) \to (C'_\bullet, \partial'_\bullet)$ is a map $P : C_n \to C'_{n+1}$ such that $\partial' P + P\partial = g - f$. Then by some algebraic manipulations,

chain homotopic maps induces the same homomorphisms between homology groups.

Actually, the (topological) homotopy between (continuous) maps induces a chain homotopy between their induced chain maps.

1.2.2 Calculation of Singular Homology

For any subspace A of X, we can define $C_n(X, A) = C_n(X)/C_n(A)$ to get another chain complex $C_\bullet(X, A)$. The boundary map $\overline{\partial}$ is inherited from the singular chain complex of X, and the cycles are called the **relative cycles** (because they are *relative to* A: the boundary of a relative cycle lies in $C_n(A)$). Its homology groups, $H_n(X, A)$, are called the **relative homology groups**.

Relative homology groups also have functoriality and homotopy invariance. Also, if X is path-connected and $*$ is a point in X, then $H_n(X, *) \cong \tilde{H}_n(X)$.

PROP 1.6 l.e.s. of a pair

A short exact sequence (s.e.s.) of chain complexes induces a long exact sequence (l.e.s.) of their homology groups:

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d] & B_n \arrow[r, "j"] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

$$\rightsquigarrow$$

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

In particular, there is a l.e.s. about the relative homology groups:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

THM 1.7 excision

Suppose $Z \subseteq A \subseteq X$ such that the closure of Z is contained inside the interior of A. Then

$$H_n(X,A) \cong H_n(X \setminus Z, A \setminus Z).$$

A pair (X,A) of a topological space X and a subspace A is called a **good pair**, if A is closed, non-empty, and *A is a deformation retract of a neighbourhood of itself*. Using the excision theorem and the l.e.s. of a pair, we can deduce that, for a good pair (X,A),

$$H_n(X,A) \cong H_n(X/A).$$

Therefore:

COR 1.8

For a good pair (X,A), there is a l.e.s. of the homology groups:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X/A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

This sequence calculates the homology of space X using a subspace A and its quotient X/A. There is another sequence, called the **Mayer-Vietoris sequence**, that calculates X by covering it with two subspaces.

THM 1.9 Mayer-Vietoris sequence

Suppose $A, B \subseteq X$ such that the *interior* of A and B cover X. Then there is a l.e.s.

$$\cdots \to H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \cap H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

where $\Phi(\alpha) = (\alpha, -\alpha)$ and $\Psi(\alpha, \beta) = \alpha + \beta$, ∂ is the (quotiented) boundary map.

With the tool of l.e.s. of a pair and excision, we can finally prove the *equivalence of singular and simplicial homology*:

$$\text{If } X \text{ is a } \Delta\text{-complex, then } H_n^\Delta(X) \cong H_n(X) \text{ for all } n.$$

1.3 Applications of Homology Theory

1.3.1 Degree

Suppose $f : \mathbb{S}^n \to \mathbb{S}^n$ $(n > 0)$ is a continuous map. Then f induces a homomorphism $f_* : H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$. $H_n(\mathbb{S}^n)$ is isomorphic to $\mathbb{Z}$, so f_* must be of the form $x \mapsto dx$ where x is a generator of $H_n(\mathbb{S}^n) \cong \mathbb{Z}$ and $d \in \mathbb{Z}$. This d is called the **degree** of the map f, $\deg f$.

Below we list some properties of the degree:

1. If f is not surjective, then $\deg f = 0$.
2. If f and g are homotopic, then $\deg f = \deg g$.
3. $\deg(f \circ g) = \deg f \cdot \deg g$. In particular, if f is a homeomorphism, then $\deg f = \pm 1$.
4. If f is a reflection, then $\deg f = -1$. Therefore, if f is the antipodal map, then $\deg f = (-1)^{n+1}$.
5. The degree of the *suspension* of f, $Sf : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$, is equal to the degree of f.

For some applications of the degree, we mention:

- If $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed point, then $\deg f = (-1)^{n+1}$.
- *(Hairy ball theorem)* A non-vanishing continuous vector field exists on $\mathbb{S}^n$, only if n is odd.
- If n is even, then $\mathbb{Z}_2$ is the only non-trivial group that can act freely on $\mathbb{S}^n$.

A powerful tool to calculate degree is the ontion of local degree. We need an additional (very weak) condition for f: there is a point y with finitely many preimages $x_1, \cdots, x_k$. Suppose U_i are disjoint neighbourhoods of x_i, V a neighbourhood of y, such that $f(U_i) \subseteq V$. Then $f(U_i - x_i) \subseteq V - y$, and we have a diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V-y) \arrow[d, "\cong"] \\
H_n(\mathbb{S}^n, \mathbb{S}^n - x_i) & H_n(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(\mathbb{S}^n, \mathbb{S}^n - y) \\
 & H_n(\mathbb{S}^n) \arrow[u, "j"] \arrow[ul, "\cong"] \arrow[r, "f_*"] & H_n(\mathbb{S}^n) \arrow[u, "\cong"']
\end{tikzcd}$$

Where k_i, j and p_i are induced by inclusions, so that the diagram commutes. The upper two isomorphisms are from excision, and the lower two isomorphisms are from the l.e.s. Focus on the top map $f_* : H_n(U_i, U_i - x_i) \to H_n(V, V - y)$. By the isomorphisms, this is also a map $\mathbb{Z} \to \mathbb{Z}$, and therefore has the form $1 \mapsto d_i$. This d_i is called the **local degree** of f at x_i, denoted $\deg f|_{x_i}$.

THM 1.10 The degree of f is equal to the sum of local degrees at each x_i. In symbols

$$\deg f = \sum_{x\in f^{-1}(y)} \deg f|_x.$$

1.3.2 Cellular Homology

A **CW complex** X is a space built inductively as follows: start from X^0, a set of discrete set. To build X^n from X^{n-1}, we attach some disks $\mathbb{D}^n$ onto X^{n-1}, by a family of **attaching maps** $\varphi_\alpha : \partial\mathbb{D}^n = \mathbb{S}^{n-1} \to X^{n-1}$. The attached disks are called the n**-cells**, and the maps $\Phi_\alpha : \mathbb{D}^n \to X^n$ are called the **characteristic maps**. We thus get a chain of spaces $X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$. We then define $X = \bigcup_{n=0}^{+\infty} X^n$, and give it the **weak topology**, i.e. $A \subseteq X$ is closed if and only if $A \cap X^n$ is closed in X^n for every n. X^n is called the n**-skeleton** of the CW complex X.

An important property of CW complexes is

PROP 1.11 A compact set in a CW complex only intersects with finitely many cells.

In particular a compact set in a CW complex is always contained in a finite skeleton.

We first state some preliminary facts:

LEM 1.12 Suppose X is a CW complex.

1. $H_k(X^n, X^{n-1})$ is zero for $k \neq n$.
2. $H_n(X^n, X^{n-1}) \cong C_n(X^n, X^{n-1}) = \mathbb{Z}\{n\text{-cells of } X\}$.
3. $H_k(X^n) = 0$ for $k > n$.
4. The map $H_k(X^n) \to H_k(X)$, induced by inclusion, is an isomorphism if $k < n$ and is surjective if $k = n$.

Now we can fit portions of the three l.e.s. of pairs (X^{n+1}, X^n), (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}) into a diagram

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"'] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

where $d_n = j_{n-1}\partial_n$ is just the "relativized" boundary map (note that $H_n(X^n, X^{n-1}) \cong C_n(X^n, X^{n-1})$, which is the free abelian group generated by the n-cells). The d_n carries information about how $(n+1)$-cells are attached to the n-cells of X^n. Using relative chains, we can ignore "holes" that comes from an earlier stage of construction, and only focus on n-cycles added in the n-skeleton.

The chain $H_n(X^n, X^{n-1})$ together with d_n makes a chain complex, the **cellular chain complex**, denoted $(C_\bullet^{\mathrm{CW}}(X), d)$, and its homology groups are called the **cellular homology groups** $H_\bullet^{\mathrm{CW}}(X)$.

A diagram chasing shows that the cellular homology group is isomorphic to the singular homology group:

$$H_n^{\mathrm{CW}}(X) \cong H_n(X).$$

This has some implications:

- If X has no n-cells, then $H_n(X) = 0$.
- More generally, if X has only finitely many n-cells, then the free rank of $H_n(X)$, $\operatorname{rk} H_n(X)$ is less than or equal to the number of n-cells in X.
- If X has no two cells in adjacent dimensions, then $H_n(X) = C_n^{\mathrm{CW}}(X) = \mathbb{Z}\{n\text{-cells}\}$. This condition is satisfied by e.g. $\mathbb{C}\mathrm{P}^n$.

To compute the cellular homology we need a more precise formula for d_n:

THM 1.13 formula of d_n

Suppose e_α^n is an n-cell of X, e_β^{n-1} are all $(n-1)$-cells of X and φ_α is the attaching map for e_α^n, i.e. $\varphi : \mathbb{S}_\alpha^{n-1} \to X^{n-1}$. By identifying everything except (the interior of) e_β^{n-1} in X^{n-1}, we will obtain $\mathbb{D}_\beta^{n-1}$ with boundary identified, i.e. $\mathbb{S}_\beta^{n-1}$. We therefore get a map

$$\mathbb{S}_\alpha^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{\text{quotient}} X^{n-1}/\left(X^{n-1} \setminus e_\beta^{n-1}\right) \cong \mathbb{S}_\beta^{n-1}$$

and this is a map from $\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$. Denote the degree of this map by $d_{\alpha\beta}$. Then

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}.$$

Intuitively, $d_{\alpha\beta}$ counts how many times the boundary of e_α^n traversed through e_β^n.

Here is a minor technicality, that the orientation of the two $(n-1)$-spheres may not be aligned (i.e. the choice for the generator of $H_n(\mathbb{S}^n)$ may not be consistent). We can do this by fixing an orientation for $\mathbb{S}^1$ and suspend it to get a fixed set of generators for each $H_n(\mathbb{S}^n)$.

1.3.3 Euler Characteristic

As an application of cellular homology, let's define the **Euler characteristic** $\chi(X)$ for a finite CW complex X:

$$\chi(X) = \sum_{i=0}^{+\infty} (-1)^n (\# \text{ of } n\text{-cells}).$$

Notice that the number of n-cells is equal to $\operatorname{rk} C_n^{\mathrm{CW}}(X)$. Using the short exact sequences

$$0 \to \ker d_n \to C_n^{\mathrm{CW}} \to \operatorname{im} d_{n+1} \to 0,$$
$$0 \to \operatorname{im} d_{n+1} \to \ker d_n \to H_n^{\mathrm{CW}} \to 0,$$

we deduce that the rank of the central term is the sum of the rank of the other two terms. Therefore, by some arithmetics,

$$\sum_n (-1)^n \operatorname{rk} C_n^{\mathrm{CW}} = \sum_n (-1)^k \operatorname{rk} H_n^{\mathrm{CW}}.$$

Therefore,

THM 1.14
Euler characteristic

$$\chi(X) = \sum_{i=0}^{+\infty} (-1)^n (\# \text{ of } n\text{-cells}) = \sum_{n=0}^{+\infty} (-1)^n \operatorname{rk} H_n(X).$$

This shows that the Euler characteristic is a topological invariant, independent of the CW structure given.

1.3.4 Lefschetz Fixed Point Theorem

The above algebraic calculations can be generalized. Using exactly the same ideas, we can deduce the

LEM 1.15
Hopf trace formula

Suppose $(C_\bullet, \partial)$ is a finite chain complex, such that every $C_\bullet$ is a free abelian group of finite rank. Suppose f is a chain map from $C_\bullet$ to itself. Then for every n, we can write down the matrix of $f : C_n \to C_n$, and compute its trace $\operatorname{tr}(f : C_n \to C_n)$. Also, f will induce maps $H_n \to H_n$; if we ignore the torsion and consider only the free part, then we can also calculate the trace $\operatorname{tr}(f_* : H_n \to H_n)$. Then

$$\sum_n (-1)^n \operatorname{tr}(f : C_n \to C_n) = \sum_n (-1)^n \operatorname{tr}(f_* : H_n \to H_n).$$

This, together with the *simplicial approximation theorem*, has an important implication, a generalization of Brouwer's fixed point theorem:

THM 1.16
Lefschetz fixed point theorem

Suppose X is a finite simplicial complex, and $f : X \to X$. We define the **Lefschetz number**

$$\Lambda(f) = \sum_n (-1)^n \operatorname{tr}(f_* : H_n(X) \to H_n(X)),$$

where we ignore the torsion part when calculating trace.

If $\Lambda(f) \neq 0$, then f must have a fixed point.

actually, the requirement of finite simplicial complex is not substantial. Three conditions are needed: compact, locally contractible, and can be embedded into some Euclidean space.

1.3.5 Homology with Coefficients

When defining the chain groups, we can replace $\mathbb{Z}$ by any abelian group G, and define $C_n(X;G) = \bigoplus_\sigma G = G\{\text{singular } n\text{-simplices}\}$. The homology groups of this chain complex, $H_n(X;G)$, is called the homology **with coefficients in** G. Theorems like excision, l.e.s. of a pair, Mayer-Vietoris sequences, etc. also apply to this homology group.

However, there should be care when dealing with degrees. Because G is generally not generated by one element, we can no longer talk about the degree of a map. There are exceptional cases, that is, when $G = k$ is a field. In this case, actually $H_n(X;k) = H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} k$.

To answer the general question about the relation of $H_n(X;G)$ and $H_n(X) = H_n(X;\mathbb{Z})$, we need to use the *universal coefficients theorem*, introduced in Section 2.2.

1.4 Axiomatic Description of Homology Theory

We now present five axioms, the *Eilenberg-Steenrod axioms*, that defines the behaviour of homology groups.

A **homology theory** is a sequence of functors H_n from the category of pairs of topological spaces (X, A) to the category abelian groups, together with a natural transformation $\partial : H_n(X, A) \to H_{n-1}(A) := H_{n-1}(A, \emptyset)$, satisfying the following axioms:

1. *homotopy invariance*: Theorem 1.5, holds.
2. *excision*: Theorem 1.7 holds.
3. *l.e.s. of a pair*: Proposition 1.6 holds.
4. *additivity*: $H_n\left(\bigsqcup_\alpha X_\alpha\right) = \bigoplus_\alpha H_n(X_\alpha)$.
5. *dimension*: $H_n(\text{a single point}) = \begin{cases} \mathbb{Z}, & n = 0; \\ 0, & n > 0. \end{cases}$

The important theorem is,

All homology theories are isomorphic.

This is just a recap of what we've learned; we can first use 4 and 5 to calculate $H_n(\mathbb{S}^0)$, then by 1, 2 and 3 calculate $H_n(\mathbb{S}^m)$, then by 3 calculate the cellular homology groups. We will use the notion of *weak equivalence* to prove the final step that the calculation of the homology of any space reduces to that of some CW complex.

2

Section 2

Cohomology Theory

2.1 Cohomology

Suppose G is an abelian group. If we apply a *contravariant Hom functor* $\mathrm{Hom}(-, G)$ to the chain complex of X, $(C_\bullet(X), \partial)$, we get another chain of maps, but reversed by contravariance:

$$\cdots \xleftarrow{\delta_{n+2}} C^{n+1}(X;G) \xleftarrow{\delta_{n+1}} C^n(X;G) \xleftarrow{\delta_n} \cdots \xleftarrow{\delta_1} C^0(X;G) \xleftarrow{\delta_0} 0,$$

where $C^n(X;G) = \mathrm{Hom}(C_n(X), G)$ and $\delta_n = \mathrm{Hom}(\partial_n, G) = - \circ \partial_n$. We also have $\delta_{n+1}\delta_n = 0$ in this case.

DEF 2.1 cohomology

This chain of maps is called the **cochain complex** of X, and its homology groups

$$H^n(X;G) = \ker \delta_{n+1} / \operatorname{im} \delta_n$$

is then called the **cohomology groups** of X.

Cohomology groups share many properties with homology groups. To name a few:

1. we have the *reduced cohomology group*, by extending the cochain complex by $\mathrm{Hom}(\mathbb{Z}, G) \cong G$;
2. we have the relative cohomology $H_n(X, A; G)$, and the related l.e.s. Notice that although Hom itself may not be exact, Hom applied to *free* abelian groups $C_n(X)$ *is* exact;
3. excision and Mayer-Vietoris sequence;
4. simplicial cohomology and cellular cohomology. Note that we will have two (equivalent) definition for the cellular cochain complex, $C^n_{\mathrm{CW}}(X) = H^n(X^n, X^{n-1})$ or $C^n_{\mathrm{CW}}(X) = \mathrm{Hom}(H^n(X^n, X^{n-1}), G)$.

2.2 Universal Coefficient Theorems and the Künneth Formulas

A natural question to ask is: how are homology groups and cohomology groups related? It turns out that there is a completely algebraic answer, relating $H^\bullet(C;G) = H_\bullet(\mathrm{Hom}(C,G))$ and $H_\bullet(C)$.

A guess might be that $H^n(C;G) \cong \mathrm{Hom}(H_n(C),G)$. This turns out to be a good guess in some cases, but it's generally not true. Actually, $\mathrm{Hom}(H_n(C),G)$ is always a direct summand of $H^n(C;G)$.

Another natural question to ask is to relate $H_n(C;G) \coloneqq H_n(C \otimes G)$ with $H_n(C) \otimes G$. Unsurprisingly, it turns out that $H_n(C) \otimes G$ is always a direct summand of $H_n(C;G)$.

The precise relation is written in the following form, called the **universal coefficient theorems** for Hom and $\otimes$:

THM 2.2 universal coefficient theorems

Suppose C is a chain complex of *free* abelian groups and G is an abelian group. Then there are natural s.e.s's

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C),G) \arrow[r] & H^n(C;G) \arrow[r] & \mathrm{Hom}(H_n(C),G) \arrow[l, dashed, bend right] \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & H_n(C) \otimes G \arrow[r] & H_n(C;G) \arrow[l, dashed, bend right] \arrow[r] & \mathrm{Tor}(H_{n-1}(C),G) \arrow[r] & 0 \end{tikzcd}$$

such that these s.e.s's split, although not naturally.

We next illustrate how to calculate Ext and Tor.

Calculating Ext
- $\mathrm{Ext}(H_1 \oplus H_2, G) = \mathrm{Ext}(H_1,G) \oplus \mathrm{Ext}(H_2,G)$.
- $\mathrm{Ext}(H,G) = 0$ if H is free.
- $\mathrm{Ext}(\mathbb{Z}_n, G) = G/nG$.

Calculating Tor
- $\mathrm{Tor}(A,B) = \mathrm{Tor}(B,A)$.
- $\mathrm{Tor}(\bigoplus A_i, B) = \bigoplus \mathrm{Tor}(A_i,B)$.
- $\mathrm{Tor}(A,B) = 0$ if A or B is free.
- $\mathrm{Tor}(\mathbb{Z}_n, A) = \ker(A \to A, a \mapsto na)$.

We will not go deeper into the detailed definition of Ext and Tor here.

The universal coefficient theorems has a powerful generalization, called the *Künneth formula*:

THM 2.3 Künneth formula

Let R be a principal ideal domain, C and C' are chain complexes of R-modules and C_n are all free modules. Then there are natural s.e.s's

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \prod_{p\in\mathbb{Z}} \operatorname{Ext}^1_R\big(H_pC, H_{p+n-1}C'\big) \arrow[r] & H_n(\operatorname{Hom}_R(C,C')) \arrow[r] & \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R\big(H_pC, H_{p+n}C'\big) \arrow[r] \arrow[l, dashed, bend right] & 0 \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \bigoplus_{p\in\mathbb{Z}} \big(H_pC \otimes_R H_{n-p}C'\big) \arrow[r] & H_n(C \otimes_R C') \arrow[r] \arrow[l, dashed, bend right] & \bigoplus_{p\in\mathbb{Z}} \operatorname{Tor}^R_1\big(H_pC, H_{n-p-1}C'\big) \arrow[r] & 0 \end{tikzcd}$$

that splits, although not naturally.

The differential on $C \otimes_R C'$ is defined as $D(c \otimes c') = \partial c \otimes c' + (-1)^{\deg c} c \otimes \partial' c'$ where $c \in C_{\deg c}$; the differential on $\operatorname{Hom}_R(C, C')$ is defined as $D(f) = \partial' f - (-1)^{\deg f} f \partial$, where $f : C_n \to C'_{n+\deg f}$.

Let us return to topology. Recall that the product of two CW complexes is still a CW complex, but the CW topology and the product topology might be different. Nonetheless, they are equal in finite dimensions, so they are equal on any compact subset, so it doesn't matter the homology groups. Notice that $C^{\mathrm{CW}}(X \times Y) = C^{\mathrm{CW}}(X) \otimes C^{\mathrm{CW}}(Y)$, therefore we have split s.e.s.

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \bigoplus_{p\in\mathbb{Z}} \big(H_pX \otimes_R H_{n-p}Y\big) \arrow[r] & H_n(X \times Y) \arrow[r] \arrow[l, dashed, bend right] & \bigoplus_{p\in\mathbb{Z}} \operatorname{Tor}\big(H_pX, H_{n-p-1}Y\big) \arrow[r] & 0. \end{tikzcd}$$

The same holds for any PID as the coefficient ring.

PROP 2.4

1. If $H_\bullet(X)$ are all free R-modules, then

$$H_n(X \times Y) \cong \bigoplus_{p\in\mathbb{Z}} \big(H_p(X) \otimes_R H_{n-p}(Y)\big).$$

2. If $H^\bullet(X)$ are all finitely generated free R-modules, then

$$H^n(X \times Y) \cong \bigoplus_{p\in\mathbb{Z}} (H^p(X) \otimes_R H^{n-p}(Y)),$$

   by the universal coefficient theorem.

We will see later, that this holds for all topological spaces, using a product called the *cross product* $\times$.

2.3 Cup Product and Cross Product

Suppose R is a ring and X a topological space. A good thing about cohomology is that we can define a *multiplication* on it, making it a ring. One such definition is called the *cup product*.

Suppose $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$. This means that $\varphi : C_k(X) \to R$ and $\psi : C_l(X) \to R$. We then define the **cup product** $\varphi \smile \psi \in C^{k+l}(X; R)$, by specifying its effect on a singular $(k+l)$-simplex σ:

$$(\varphi \smile \psi)(\sigma) := \varphi\Big(\sigma|_{[v_0,\cdots,v_k]}\Big) \cdot \psi\Big(\sigma|_{[v_k,\cdots,v_l]}\Big).$$

We can check that $\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^k \varphi \smile (\delta\psi)$. Therefore, the cup product of two cocycles is a cocycle, and the cup product of a coboundary and a cocycle is a coboundary. Therefore, $\smile : C^k \times C^l \to C^{k+l}$ induces a well-defined map $H^k \times H^l \to H^{k+l}$, making $H^\bullet(X;R)$ a ***graded** R**-algebra***.

There is also a relative version: if A and B are open sets in X, or subcomplexes of a CW complex X, then we can define a cup product

$$\smile : H^k(X,A;R) \times H^l(X,B;R) \to H^{k+l}(X, A \cup B; R).$$

Also, if $f : X \to Y$ is a map between topological spaces, then it induces ring homomorphisms: $f_* : H^\bullet(X;R) \to H^\bullet(Y;R)$.

Another important property of this ring structure is its ***graded commutativity***. That is, if $\alpha \in H^k(X;R)$ and $\beta \in H^l(X;R)$, then $\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha$.

Suppose X and Y are topological spaces and $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projection maps. Then for any $\alpha \in H^\bullet(X;R)$ and $\beta \in H^\bullet(Y;R)$, we can pull them back to get cochains in $H^\bullet(X \times Y)$: $p_1^*\alpha$ and $p_2^*\beta$. We define the **cross product** $\alpha \times \beta = p_1^*\alpha \smile p_2^*\beta$. Since this cross product is clearly R-bilinear, it actually induces a map

$$\times : H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \to H^\bullet(X \times Y;R).$$

If X and Y are CW complexes, then this is a map we have seen before: the map appeared in the Künneth formula (Proposition 2.4). In particular, if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module for every k, then $\times$ is an R-module isomorphism.

We can make $\times$ a *ring* homomorphism, by defining an appropriate ring structure on $H^\bullet(X;R) \otimes_R H^\bullet(Y;R)$, as follows: $(a \otimes b) \cdot (c \otimes d) := (-1)^{\deg b \cdot \deg c} (a \smile c) \otimes (b \smile d)$.

We also have the relative version of cross product:

$$\times : H^\bullet(X,A;R) \otimes_R H^\bullet(Y,B;R) \to H^\bullet(X \times Y, (A \times Y) \cup (X \times B); R)$$

and the reduced version:

$$\tilde{H}^\bullet(X;R) \otimes_R \tilde{H}^\bullet(Y;R) \to \tilde{H}^\bullet(X \wedge Y;R)$$

where $X \wedge Y := (X \times Y)/(X \times \{y_0\} \cup \{x_0\} \times Y)$ is the smash product with base-points $x_0 \in X$ and $y_0 \in Y$.

As a special case, if $X = Y$, we then have $H^\bullet(X;R) \otimes_R H^\bullet(X;R) \xrightarrow{\times} H^\bullet(X \times X) \xrightarrow{\Delta^*} H^\bullet(X)$ where $\Delta : X \to X \times X, x \mapsto (x,x)$. The composition of these two maps is exactly the cup product $\smile$.

2.4 Poincaré Duality

Poincaré duality is another fundamentally different way of relating cohomology groups with homology groups of *topological manifolds*. Its general form states that,

THM 2.5 Poincaré duality

Suppose M is an n-dimensional compact connected manifold, then

1. $H_k(M;\mathbb{Z}_2) \cong H^k(M;\mathbb{Z}_2)$ for every $0 \leqslant k \leqslant n$;
2. if we further assume that M is orientable, then $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$ for every $0 \leqslant k \leqslant n$.

Our next subsections will be devoted to proving this deep result, and discussing its significance on calculating the cup product structure.

2.4.1 Orientation

In this section and sections to follow, we mainly consider connected, closed, compact manifold.

Suppose U is a subset of M. We define the **local homology group** of U, denoted by $H_k(M \mid U)$, by $H_k(M, M \setminus U)$.

If $A \subseteq B$, then $M \setminus B \subseteq M \setminus A$. Therefore, any cycle relative to $M \setminus B$ is a cycle relative to $M \setminus A$, so we have a *restriction map* $\cdot|_A : H_k(M \mid B) \to M_k(M \mid A)$, induced by the *inclusion* $M \setminus B \hookrightarrow M \setminus A$.

If $U = \{x\}$ is a point, or a contractible open neighbourhood of a point that is small enough, then by excision $H_n(M \mid U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$. Intuitively, it describes how many times a chain, with boundary in $M \setminus U$, wraps around the "hole" U.

There are two choices for the generator of $H_n(M \mid U)$. A chosen generator for $H_n(M \mid U)$ is called the a **local orientation** μ_U.

We now choose a local orientation $\mu_x \in H_n(M \mid x)$ for every $x \in M$. We say that they satisfy the **local consistency condition**, that is, for every $x \in M$, there is a small open set $U \cong \mathbb{R}^n$ around x and a local orientation μ_U of U, such that for any $y \in U$, $\mu_U|_y = \mu_y$, i.e. the restriction map $H_n(M \mid U) \to H_n(M \mid y)$ takes μ_U to μ_y.

Intuitively, we draw a small loop μ_x around every point x, and the local consistency condition says that the loops goes in the same direction in the proximity of every point.

If M admits a choice of the local orientations μ_x that satisfies the local consistency condition, we say that M is **orientable**. Otherwise, say M is **non-orientable**.

Every non-orientable manifold M has a double covering $\widetilde{M}$. This $\widetilde{M}$ can be constructed as follows: as a set, $\widetilde{M} = \{(x, \mu_x) : x \in M, \mu_x \text{ a local orientation}\}$. Since for each x, there are two choices for μ_x, so $\widetilde{M} \to M$ is a two-to-one map. We give $\widetilde{M}$ a topology, generated by sets $\tilde{U} = \{(x, \mu_x) : x \in U, \mu_x \text{ a local orientation consistent with } \mu_U\}$ for sufficiently small sets $\mathbb{R}^n \cong U \subseteq M$, making $\widetilde{M}$ a covering space of M. This constructed $\widetilde{M}$ is always orientable.

This kind of construction can be done to any manifold, not just non-orientable ones. When one do it on an orientable manifold, what one get is a trivial covering map, with $\widetilde{M}$ being just a disjoint union of two copies of M. But when one do it on a non-orientable manifold, there is some loop on M that when one goes around it once the orientation reverses, as on a Möbius band. Therefore, $\widetilde{M}$ will be a connected manifold, and there is no way to find a copy of M in it. In other words, there is no section $M \to \widetilde{M}$.

In light of this observation, we introduce the notion of orientability to arbitrary coefficient rings. Suppose R is a ring. We similarly define a manifold, a covering space of M, by $M_R = \{(x, \alpha_x) : x \in M, \alpha_x \in H_n(M \mid x; R)\}$, with topological basis $U_R = \{(x, \alpha_x) : x \in U, \alpha_x \in H_n(M \mid x; R) \text{ such that } \alpha_U|_x = \alpha_x\}$ for any chosen U and some fixed $\alpha_U \in H_n(M \mid U; R)$. The covering map is given by $M_R \to M, (x, \alpha_x) \mapsto x$, so the fiber of each $x \in M$ is R with discrete topology.

Since $H_n(M \mid x; R) \cong H_n(M \mid X; \mathbb{Z}) \otimes_{\mathbb{Z}} R$, we can consider a subspace $M_r = \{(x, \pm\mu_x \otimes r)\} \subseteq M_R$, where μ_x is a generator of $H_n(M \mid x; \mathbb{Z})$. If $r = -r$, then $M_r = M_{-r} = M$. If $r \neq -r$, then $M_r = M_{-r}$ will be the double cover $\widetilde{M}$ we defined before. Therefore, M_R is the disjoint union $\bigsqcup_{r \in R} M_r$, and each M_r is either M or $\widetilde{M}$. For example, $M_{\mathbb{Z}}$ is the disjoint union of one M (corresponding to $r = 0$) and one $\widetilde{M}$ for each $r \in \mathbb{Z}_+$.

We have seen that M is orientable if and only if we can find a copy of M in $\widetilde{M}$, or a section $M \to \widetilde{M}$. Similarly we define the notion of R-orientability. An **R-orientation** is a section $M \to M_R, x \mapsto \alpha_x$, such that α_x is a generator (equivalently, a unit) for $H_n(M \mid x; R)$. A manifold that admits an R-orientation is called **R-orientable**, and a manifold with a fixed R-orientation is called an **R-oriented manifold**.

If M is orientable, i.e. $\mathbb{Z}$-orientable, then it is R-orientable for every ring R. If M is not orientable, then it's R-orientable if and only if there is a unit $u \in R$ such that $2u = 0$. In particular, every manifold is $\mathbb{Z}_2$-orientable.

2.4.2 The Fundamental Class

We will establish the following result:

THM 2.6 Let M be a closed, compact, connected n-dimensional manifold, and $x \in M$ a point.

1. If M is R-orientable, then the map $\cdot|_x : H_n(M;R) \to H_n(M\,|\,x;R)$ is an (R-module) isomorphism.

2. If M is not R-orientable, then the previously mentioned map is injective, with image $\{r \in R : 2r = 0\}$.

3. For $i > n$, $H_i(M;R) = 0$.

The technical part is the following lemma:

LEM 2.7 Suppose M is any n-dimensional manifold, and $A \subseteq M$ a compact subset.

1. If $x \mapsto \alpha_x$ is a section of $M_R \to M$, then there is a unique $\alpha_A \in H_n(M\,|\,A;R)$ such that $\alpha_A|_x = \alpha_x$.

2. $H_i(M\,|\,A;R) = 0$, for any $i > n$.

of Theorem. Assuming this lemma is true. Denote by $\Gamma_R(M)$ the R-module of sections of $M_R \to M$. Consider

$$\begin{aligned}\Phi : H_n(M;R) &\to \Gamma_R(M),\\ \alpha &\mapsto (x \mapsto \alpha|_x).\end{aligned}$$

This is an R-linear map. The Lemma tells us that Φ is an isomorphism by taking $A = M$ (notice that $H_k(M\,|\,M) = H_k\big(M,\emptyset\big) = H_k(M)$).

By uniqueness of path lifting and M is connected, we know that every section in $\Gamma_R(M)$ is actually uniquely determined by its value at one point in M. Therefore, fix a point x, consider the evaluation map

$$\begin{aligned}\mathrm{ev} : \Gamma_R(M) &\to H_n(M\,|\,x;R),\\ s &\mapsto s(x).\end{aligned}$$

Then ev is injective. Therefore, considering the diagram,

$$\begin{tikzcd} H_n(M;R) \arrow[r, "\Phi", "\cong"'] \arrow[rr, bend right, "\cdot|_x"'] & \Gamma_R(M) \arrow[r, hook, "\mathrm{ev}"] & H_n(M\,|\,x;R) \end{tikzcd}$$

Since ev is injective and Φ is an isomorphism, $\cdot|_x$ is injective and its image is equal to the image of ev.

If M is R-orientable, then $\cdot|_x$ is an isomorphism since its image contains a generator, the one making M R-orientable.

If M is not R-orientable, actually, if M is not $\mathbb{Z}$-orientable, then $M_R = \bigsqcup_r M_r$ where $M_r = M$ if $2r = 0$, $M_r = \widetilde{M}$ if $2r \neq 0$. Since M is not orientable, there is no section from M to $\widetilde{M}$; therefore, every section is from M to some M_r where $2r = 0$. Therefore, $\operatorname{im} \cdot|_x = \operatorname{im} \mathrm{ev} = \{r \in R : 2r = 0\}$. □

of Lemma. We start from simple sets, and gradually building up to general compact set A.

Step 1. We claim that, if the Lemma holds for A, B and $A \cap B$, then it also holds for $A \cup B$. This can be proved by looking into the Mayer-Vietoris sequence

$$0 \to H_n(M \mid A \cup B) \xrightarrow{\Phi} H_n(M \mid A) \oplus H_n(M \mid B) \xrightarrow{\Psi} H_n(M \mid A \cap B) \to \cdots$$

Step 2. For any compact set A, we can write A as the union of finitely many smaller compact sets, each is contained inside a single local chart of M. Using step 1 and induction on the number of smaller sets, we only need to consider the case of *one* compact set contained in a single local chart. And by using excision, we reduced M to $\mathbb{R}^n$.

Step 3. If A is a convex set then A deformation retracts to any point $x \in A$ and $\mathbb{R}^n \setminus A$ deformation retracts to an $(n-1)$-sphere around x. Therefore we have a map

$$\begin{gathered} H_i(\mathbb{R}^n \mid A) \to H_i(\mathbb{R}^n \mid x), \\ \alpha_A \mapsto \alpha_A|_x. \end{gathered}$$

Therefore, statement 1 holds for convex A, and therefore for finite unions of convex sets A.

Step 4. For general A, consider $\alpha = [z] \in H_i(\mathbb{R}^n \mid A)$, suppose $\partial z = \sum_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^{i-1} \to \mathbb{R}^n \setminus A$. Let $C = \bigcup_\alpha \operatorname{im} \sigma_\alpha$, then C is a compact set. Cover A with finitely many closed balls B_i disjoint from C, and let K be the union of these closed balls. Then, from step 1 and 3, statement 1 holds for K, so $z \in C_i(\mathbb{R}^n \mid K)$. Therefore, $\cdot\,|_A : H_i(\mathbb{R}^n \mid K) \to H_i(\mathbb{R}^n \mid A)$ sends $[z]$ to $[z]$. If $H_i(\mathbb{R}^n \mid K) = 0$, then $H_i(\mathbb{R}^n \mid A) = 0$, so statement 2 holds.

Now we prove statement 1.

Existence. Take any ball $B \subseteq A$. Then $\alpha_A := \alpha_B|_A$ satisfies the conditions.

Uniqueness. The cycle z defines an element $\alpha_K \in H_n(M \mid K)$, which $\alpha_K|_A = \alpha_A$. If α'_A is another element satisfying the Lemma, let $\alpha = \alpha_A - \alpha'_A$ and $\alpha_0 = \alpha_K - \alpha'_K$ where α'_K defined similarly, then $\alpha|_x = 0$ for every $x \in A$. Since $H_n(M \mid B_i) \to H_i(M \mid x)$ is an isomorphism for every $x \in B_i$, $\alpha_0|_x = 0$ for every $x \in K$ since this is true for $x \in A$. Therefore, by the uniqueness part of step 3, $\alpha_0 = 0$, so $\alpha = 0$ i.e. $\alpha_A = \alpha'_A$. $\square$

This theorem has some important implications.

First, it implies that if M is R-orientable, then $H_n(M; R) \cong H_n(M \mid x; R) \cong R \cong H_0(M; R)$, which is the $k = n$ case of the Poincaré duality theorem.

Second, it shows that an R-oriented manifold M determines a choice of a generator $[M] \in H_n(M; R) \cong R$. This $[M]$ is called a **fundamental class** of M. Conversely, if $H_n(M; R) \cong R$, then a fundamental class $[M]$ can be chosen, and we can define an R-orientation by sending x to $[M]|_x$. A fundamental class of an orientable manifold determines an orientation of that manifold, and vice versa.

Finally, from the proof and the universal coefficient theorem, we can get some information about the highest two homology groups from orientability:

For the highest homology group, if we take $\mathbb{Z}$ coefficients, then $H_n(M) \cong \mathbb{Z}$ if M is orientable, and $H_n(M) \cong 0$ if M is non-orientable.

For the second highest, the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable, and $\mathbb{Z}_2$ if M is non-orientable.

2.4.3 Compactly Supported Cohomology

Our proof of the Poincaré duality follows a similar route as that of Lemma 2.7. However we will encounter a difficulty in the last step, where we reduce to the case $M = \mathbb{R}^n$; Poincaré duality does *not* hold for non-compact manifolds like $\mathbb{R}^n$. Therefore, we need an extension of the duality theorem to non-compact manifolds.

Suppose X is any space and R a commutative ring. For a cochain $\varphi \in C^i(X;G)$, we say that φ is **compactly supported** if there is a compact set $K \subseteq X$, such that for any chain $\sigma : \Delta^i \to X \setminus K$, $\varphi(\sigma) = 0$. Obviously if X is itself compact, then any cochain is compactly supported. Let $C^i_c(X;G)$ be the subgroup containing all compactly supported i-cochains. It's easy to check that the coboundary map of any compactly supported cochain is sill compactly supported. Therefore, $C^\bullet_c$ forms a cochain complex, and its homology groups are defined as the **compactly supported cohomology groups**, $H^i_c(X;R)$.

The compactly supported cohomology is *not* natural as a functor from Top to AbGp; rather, it is only natural under *proper* maps. A continuous map between spaces is called **proper**, if the preimage of any compact set is compact; a homotopy $F : X \times [0,1] \to Y$ is called a **proper homotopy** if the map F is proper. Then the compactly supported cohomology groups are natural under *proper* maps and are invariant under *proper* homotopies. As an example, $H^1_c(\mathbb{R}) \cong \mathbb{Z}$ but $H^1_c(\text{a point}) = 0$.

The compactly supported cohomology can also be defined using direct limits. Suppose I is a poset, such that any two elements of I has a common upper bound. Such a poset is called a **directed set**. Given a functor $I \to \mathsf{AbGp}, \alpha \mapsto G_\alpha, (\alpha \leqslant \beta) \mapsto \left(f_{\alpha\beta} : G_\alpha \to G_\beta\right)$, we can define the **direct limit** of the groups G_α, denoted $\varinjlim_\alpha$, by $\coprod_\alpha G_\alpha \big/ \sim$, where $\sim$ is defined by $a \sim b$, where $a \in G_\alpha$ and $b \in G_\beta$, iff there is a $c \in G_\gamma$, such that $\alpha \leqslant \gamma$, $\beta \leqslant \gamma$, and a and b maps to the same element c under $f_{\alpha\gamma}$ and $f_{\beta\gamma}$. Intuitively, the direct limit is the "smallest group containing all groups G_α".

PROP 2.8 Suppose X is the union of a directed set $\{X_\alpha\}$ of topological spaces ordered by inclusion, such that every compact set K is contained entirely inside X_α. Then

- $\varinjlim_\alpha H_i(X_\alpha; G) = H_i(X; G)$.
- $\varinjlim_\alpha H^i_c(X, X \setminus X_\alpha; G) = H^i_c(X; G)$.

As a corollary, the compact subsets of X form a directed set under inclusion; therefore,

$$H^i_c(X; G) = \varinjlim_{K \text{ compact}} H^i(X, X \setminus K; G).$$

The first statement about homology holds without compactness assumptions because the images of simplices are already compact by themselves.

A particular thing about compactly supported cohomology is that it can be *covariant*, as opposed to the normal cohomology group which is contravariant. This is because we can pushforward a compactly supported cochain, as follows. Suppose $i : U \hookrightarrow V$ is an inclusion of open sets. Then for any $\varphi \in C^k_c(U)$, we can extend it to a cochain in $C^k_c(V)$, by simply defining $\varphi = 0$ outside of U. Since the support is already inside U, we get a well-defined chain. This gives us a map $i_* : C^k_c(U) \to C^k_c(V)$, and thus a map $i_* : H^k_c(U) \to H^k_c(V)$.

We can now generalize the Poincaré duality to non-compact manifolds:

THM 2.9 Poincaré duality for non-compact manifolds

Suppose M is an R-orientable n-dimensional connected manifold, no need to be compact. Then $H^k_c(M; R) \cong H_{n-k}(M)$.

To prove this we must first construct a map from $H^k_c(M; R)$ to $H_{n-k}(M)$, and then verify that it is an isomorphism. This map turns out to be constructed using *cap products*, a product that looks much like the cup product.

2.4.4 Cap Products, and the Poincaré Duality Map

Let X be any space and R be a commutative ring. We define the **cap product**

$$\begin{aligned} \frown : C_k(X; R) \times C^l(X; R) &\to C_{k-l}(X; R), \\ \sigma \frown \varphi &\mapsto \varphi\left(\sigma|_{[v_0, \cdots, v_l]}\right) \cdot \sigma|_{[v_l, \cdots, v_k]}. \end{aligned}$$

The cap product satisfies that $\partial(\sigma \frown \varphi) = (-1)^l(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$; so the cap product induces a well defined map on (co)homology groups $H_k(X; R) \times H^l(X; R) \to H_{k-l}(X; R)$. Moreover, like the cup product, the cap product is a natural map.

There is also a relative version:

$$\frown : H_k(X, A \cup B; R) \times H^l(X, A; R) \to H_{k-l}(X, B; R).$$

Let M be an R-orientable n-dimensional manifold, and fix an orientation $x \mapsto \mu_x$. Consider compact subsets $K \subseteq L$. Then the inclusion $i : K \hookrightarrow L$ induces i_* and i^* on (co)homology groups. Consider the following diagram:

$$\begin{tikzcd}
H_n(M\,|\,L) \times H^k(M\,|\,L) \arrow[d, "i_*"] \arrow[dr, "\frown"] & \\
H_n(M\,|\,K) \times H^k(M\,|\,K) \arrow[u, "i^*"'] \arrow[r, "\frown"'] & H_{n-k}(M)
\end{tikzcd}$$

Using Lemma 2.7, there is a unique $\mu_K \in H_n(M\,|\,K)$, such that $\mu_K|_x = \mu_x$ for any $x \in K$; we have the same for L. Uniqueness implies that $\mu_L|_K = i_*(\mu_L) = \mu_K$. Then naturality of the cap product implies that, for any $\varphi_K \in H^k(M\,|\,K)$, $\mu_K \frown \varphi_K = \mu_L \frown i^*(\varphi_K)$. Therefore, the map $H^k(M\,|\,K) \to H_{n-k}(M), \varphi \mapsto \mu_K \frown \varphi$ is natural under inclusion of K; therefore, it induces a well-defined map on the direct limit $\varinjlim_K H^k(M\,|\,K) \to H_{n-k}(M)$. This gives a map $D_M : H^k_c(M) \to H_{n-k}(M)$.

THM 2.10 the Poincaré duality map

The map D_M, defined as above, is an isomorphism for any connected n-dimensional manifold M.

Proof. Suppose U and V are open subsets. We have two Mayer-Vietoris sequences

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^i_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^i_cU \oplus H^i_cV \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^i_c(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & H^{i+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^{i+1}_cU \oplus H^{i+1}_cV \arrow[r] \arrow[d, "D_U \oplus -D_V"] & \cdots \\
\cdots \arrow[r] & H_{n-i}(U\cap V) \arrow[r] & H_{n-i}U \oplus H_{n-i}V \arrow[r] & H_{n-i}(U\cup V) \arrow[r] & H_{n-i-1}(U\cap V) \arrow[r] & H_{n-i-1}U \oplus H_{n-i-1}V \arrow[r] & \cdots
\end{tikzcd}$$

Notice the first row is different from the Mayer-Vietoris sequence of normal cohomology groups, because we can make compactly supported cohomology groups covariant. By the five lemma, if D_U, D_V and $D_{U\cap V}$ are all isomorphisms, then so if $D_{U\cup V}$.

We clain that, if $M = \bigcup_{i\in\mathbb{Z}} U_i$ where $U_1 \subseteq U_2 \subseteq \cdots$, and D_{U_j} are all isomorphisms, then D_M is also an isomorphism. This is seem by

$$\begin{tikzcd}
\varinjlim_j H^i_c(U_j) \arrow[r, phantom, "="] \arrow[d, "D_{U_j}"] & \varinjlim_j \varinjlim_{K\subseteq U_j} H^i(U_j\,|\,K) \arrow[r, phantom, "="] & \varinjlim_{K\subseteq M} H^i(M\,|\,K) \arrow[r, phantom, "="] & H^i_c(M) \arrow[d, "D_M"] \\
\varinjlim_j H_{n-i}(U_j) \arrow[rrr, equal] & & & H_{n-i}(M)
\end{tikzcd}$$

We can check that $D_{\mathbb{R}^n}$ is an isomorphism; since every manifold is second countable, we finished the proof. $\square$

In the case of a closed (compact) manifold M, this D_M is given by $D_M(\varphi) = [M] \frown \varphi$.

2.4.5 The Poincaré Pairing

The cap product is closely related to the cup product. Some basic identities involving both are

- $\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$ for $\alpha \in H_{k+l}$, $\varphi \in H^k$ and $\psi \in H^l$;
- $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \smile \psi)$, which makes the homology group into a right module of the cohomology ring.

Recall that by the universal coefficient theorem there is a surjective homomorphism $h : H^k \to \mathrm{Hom}_R(H_k, R)$, but this is in general not an isomorphism. If we fix a $\varphi \in H^l$, then we get a commutative diagram

$$\begin{tikzcd}
H^l \arrow[r, "h"] \arrow[d, "\varphi \smile"'] & \mathrm{Hom}_R(H_l, R) \arrow[d, "(\frown \varphi)^*"] \\
H^{l+k} \arrow[r, "h"'] & \mathrm{Hom}_R(H_{l+k}, R)
\end{tikzcd}$$

In particular, when h is an isomorphism, the cap product and the cup product are dual to each other.

From this relation we can state the *Poincaré duality for cup products*. A **bilinear pairing** on two R-modules A and B is a bilinear map $(,) : A \times B \to R$, or equivalently a map $(,) : A \otimes_R B \to R$. A bilinear pairing is called **non-singular**, if the induced maps $A \to \mathrm{Hom}_R(B, R), a \mapsto (b \mapsto (a, b))$ and $B \to \mathrm{Hom}_R(A, R), b \mapsto (a \mapsto (a, b))$ are isomorphisms.

For an R-oriented manifold M, we can define the **Poincaré pairing**:

$$\begin{aligned} H^k(M; R) \otimes_R H^{n-k}(M; R) &\to R, \\ (\varphi, \psi) &\mapsto (\varphi \smile \psi)([M]). \end{aligned}$$

THM 2.11 Poincaré pairing

If M is an R-oriented closed n-dimensional manifold, and R is a field, or in the case $R = \mathbb{Z}$ we consider only the free abelian part of cohomology groups, then this Poincaré pairing is non-singular.

By graded commutativity of the cup product, this Poincaré pairing is also graded commutative. In particular, as an application, if $n = 4k + 2$, then the pairing on $H^{2k+1}(M) \otimes H^{2k+1}(M)$ is anti-symmetric, showing that $H^{2k+1}(M)$ must have even dimension. Therefore, if M is a $(4k + 2)$-dimensional oriented compact manifold, then $\chi(M)$ is even.

2.4.6 Transversal Intersection and Cup Product

Poincaré duality also allows us to understand cup products in a geometrical way. Suppose M is an R-oriented closed connected n-dimensional manifold.

We fix the coefficient ring R, and denote $D : H^k(M) \to H_{n-k}(M)$ the duality map. Poincaré duality tells us that D is an isomorphism, so an inverse exists: $D^{-1} : H_k(M) \to H^{n-k}(M)$. For a homology class $[\alpha]$, the dual cohomology class $D^{-1}([\alpha])$ will frequently be denoted by $[\alpha]^*$.

We define the **intersection product** $a \bullet b$:

$$\begin{tikzcd}
H_i(M) \times H_j(M) \arrow[r, "\bullet"] & H_{i+j-n} \\
H^{n-i}(M) \times H^{n-j}(M) \arrow[u, "D"] \arrow[r, "\smile"'] & H^{2n-i-j}(M) \arrow[u, "D"']
\end{tikzcd}$$

In formula, $\bullet : H_i(M) \times H_j(M) \to H_{i+j-n}(M)$ is defined by

$$a \bullet b = D\big(D^{-1}(a) \smile D^{-1}(b)\big).$$

This formula can be simplified to $a \bullet b = a \frown D^{-1}(b)$.

The name "intersection product" can be justified as follows. Suppose that M is moreover a smooth manifold, and A and B are embedded oriented closed connected submanifolds of M, with dimensions $n-i$ and $n-j$ respectively. We say that A and B intersect **transversely**, if at any point $p \in A \cap B$, $\mathrm{T}_pA + \mathrm{T}_pB = \mathrm{T}_pM$. In this case, $A \cap B$ is a $n-(i+j)$-dimensional embedded submanifold, and inherits an orientation from A and B.

We spend some time here to explain how exactly we orient $A \cap B$. An orientation on a smooth manifold can be described by bases of tangent spaces; a basis of the tangent space at p is a local orientation at p, and two orientations are consistent if one can transform one to another using a linear transformation with positive determinant. We can relate the tangent spaces of A, B and $A \cap B$ using a natural s.e.s.

$$0 \to \mathrm{T}_p(A \cap B) \to \mathrm{T}_pA \oplus \mathrm{T}_pB \to \mathrm{T}_pM \to 0$$

This gives local orientations of $A \cap B$ through local orientations of A, B and M.

Inclusion gives $H_{n-i}(A) \to H_{n-i}(M)$ and $H_{n-j}(B) \to H_{n-j}(M)$, and we denote, by abuse of notation, the images of the fundamental classes in $H_\bullet(M)$ by $[A]$ and $[B]$. The main theorem is:

THM 2.12 cup product is dual to transverse intersection

For embedded oriented closed connected manifolds A and B of an oriented closed manifold M,

$$[A] \bullet [B] = [A \cap B].$$

Equivalently, in cup products,

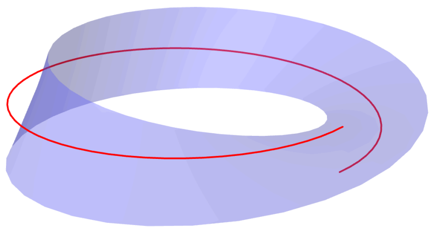
$$[A]^* \smile [B]^* = [A \cap B]^*,$$

where $[\alpha]^* = D^{-1}([\alpha])$ is the Poincaré dual of $[\alpha]$.

In general, however, cup products are more general than intersection products, as not every homology class of M is represented as the fundamental class of some submanifold.

As a corollary, since H_0 of a connected manifold is always $\mathbb{Z}$ and can only take one of the two orientations, if two submanifolds intersects at an odd number of points, then they cannot cancel out in H_0; therefore, if A and B intersects at an odd number of points, then $[A \cap B]$ is not zero, and therefore $D^{-1}(A)$ and $D^{-1}(B)$ are non-zero.

For an application, we can show using this corollary that non-orientable surfaces cannot be embedded in $\mathbb{S}^3$, because we can otherwise find a loop in $\mathbb{S}^3$ that only intersect it once, but $H_2\left(\mathbb{S}^3\right) = 0$. (As depicted in the following graph; connecting the two ends of the red arc gives such a curve.)



For an application, all hyperplanes in $\mathbb{C}\mathrm{P}^n$ (i.e. the zero set of $az_0 + bz_1 + cz_2 = 0$ for a, b and c not all zero) shares a same homology class τ where $|\tau| = 2$, and $H_\bullet(\mathbb{C}\mathrm{P}^n) = \mathbb{Z}[\tau]/(\tau^n)$. We can use this to imply the Bézout theorem for $\mathbb{C}\mathrm{P}^2$: if two smooth algebraic curves of degees d_1 and d_2 intersects transversally, then they intersect at $d_1 d_2$ points.

3

Section 3

Higher Homotopy Groups

3.1 Definition

Suppose (X, x_0) is a based space, and $I = [0, 1]$.

DEF 3.1 The n-**th homotopy group** is defined, as a set, by

$$\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \to (X, x_0)\}/(\text{homotopy rel } \partial I^n).$$

The group structure can be defined for $n \geqslant 1$, as

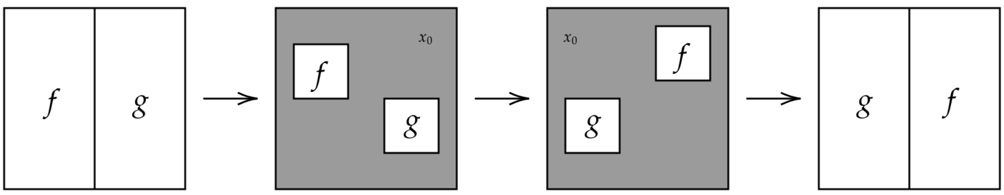
$$(f * g)(s_1, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n) \text{ if } 0 \leqslant s_1 \leqslant \frac{1}{2} \\ g(2s_1 - 1, s_2, \cdots, s_n) \text{ if } \frac{1}{2} < s_1 \leqslant 1. \end{cases}$$

In particular, $\pi_1(X, x_0)$ is the familiar fundamental group.

An interesting property of higher homotopy groups is that

PROP 3.2 For $n \geqslant 2$, $\pi_n(X, x_0)$ is always abelian.

For an intuition about why this is right, see the following diagram:



Similar to the case of fundamental groups,

PROP 3.3 If γ is a path from x_0 to x_1, then we have an isomorphism $\gamma_* : \pi_n(X, x_0) \to \pi_n(X, x_1), [f] \mapsto [\gamma f]. [\gamma f]$.

In particular, this gives us a group action $\pi_1(X, x_0) \curvearrowright \pi_n(X, x_0)$. If this action is trivial, we say that X is **n-simple**, and write $\pi_n(X)$, omitting the basepoint.

Since $I^n / \partial I^n \cong \mathbb{S}^n$, he group $\pi_n(X, x_0)$ can also be described as the set $\{f : (\mathbb{S}^n, *) \to (X, x_0)\}$, modulo homotopy rel $*$. The addition can then be described as

$$f * g : \mathbb{S}^n \xrightarrow{\text{contract the equator through } *} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{f \vee g} X.$$

As is the case for π_1, higher homotopy groups are functors

$$\mathsf{Top}_* = \{\text{based space}\} \to \{\text{abelian groups}\} = \mathsf{AbGp},$$

and homotopy equivalences induces isomorphisms.

Homotopy groups behave particularly well under covering maps:

PROP 3.4 A covering map $p : \tilde{X} \to X$ induces an isomorphism $p_* : \pi_n(\tilde{X}, \tilde{x_0}) \to \pi_n(X, x_0)$ for all $n \geqslant 2$.

In particular, if X has a contractible universal cover, then $\pi_n(X) = 0$ for $n \geqslant 2$.

The homotopy groups of products are simple to compute:

PROP 3.5

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha).$$

We can also define the **relative homotopy groups** $\pi_n(X, A, x_0)$ for $x_0 \in A \subseteq X$. Let $J^{n-1} \subseteq I^n$ be the closure of the union of all but one faces of the n-cube I^n. Then $\pi_n(X, A, x_0)$ consists of all maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, modulo homotopy through such maps. Group multiplication is defined similarly. In terms of spheres, this is the set of maps $(\mathbb{D}^n, \mathbb{S}^{n-1}, *) \to (X, A, x_0)$, modulo homotopy through such maps. The relative homotopy group is indeed a group only when $n \geqslant 2$, and is abelian when $n \geqslant 3$. The intuition is that, in the diagram for Proposition 3.2, this time we need that the rectangles f and g be "rooted" in A, and in dimension two, there is no enough place for the two rooted rectangles to swap positions, while in dimension three there is enough place.

PROP 3.6
compression criterion When $n \geqslant 2$, a map $(\mathbb{D}^n, \mathbb{S}^{n-1}, *) \to (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$ if and only if it's homotopic to a map $\mathbb{D}^n \to A$, through a homotopy of maps $(\mathbb{D}^n, \mathbb{S}^{n-1}, *) \to (X, A, x_0)$.

Like in homology groups, there is a l.e.s. of relative homotopy groups:

THM 3.7 There is a long exact sequence

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0)$$

where i_* is induced by inclusion $i : A \to X$ and j_* is induced by inclusion $j : (X, x_0, x_0) \to (X, A, x_0)$. ∂ comes from restricting a map $(\mathbb{D}^n, \mathbb{S}^{n-1}, *) \to (X, A, x_0)$ to $(\mathbb{S}^{n-1}, *) \to (A, x_0)$.

There should be special care to the last few terms; they are not groups. However, as sets, they have a special element, namely, the homotopy class represented by constant maps. For maps involving these sets, kernel means every element that get mapped to the homotopy class of constant maps.

A based space (X, x_0) is called n**-connected** if $\pi_i(X, x_0) = 0$ for all $i \leqslant n$. Then the following statements are equivalent:

1. X is n-connected for every basepoint.
2. every continuous map $\mathbb{S}^i \to X$ is nullhomotopic, for all $i \leqslant n$.
3. every $\mathbb{S}^i \to X$ can be extended to $\mathbb{D}^{i+1} \to X$ for all $i \leqslant n$.

3.2 Three Key Results

There are three key results in the study of homotopy groups: *cellular approximation*, *Whitehead's theorem* and *CW approximation*.

3.2.1 Cellular Approximation

Suppose X and Y are CW complexes. A **cellular map** $f : X \to Y$ is such a map that maps the n-skeleton of X to the n-skeleton of Y, for every n.

THM 3.8 cellular approximation theorem

Every continuous map between CW complexes is homotopic to a cellular map.

As an example, let $\mathbb{S}^k = e^0 \cup e^k$. For any $f : \mathbb{S}^n \to \mathbb{S}^k$, f can be homotoped to a cellular map. If $n < k$, then $\operatorname{im} f$ is a point (since the n-skeleton of $\mathbb{S}^k$ is just a point). Therefore, f is a constant map. We have just proved that

$$\pi_n\left(\mathbb{S}^k\right) = 0 \text{ for } n < k.$$

As a corollary

COR 3.9

A CW pair (X, A) is n-connected if all cells in $X \setminus A$ has dimension larger than n.

In particular, (X, X^n) is n-connected, where X^n is the n-skeleton of X.

Therefore, the inclusion $X^n \hookrightarrow X$ induces isomorphisms for π_i for $i < n$:

COR 3.10

The i-th homotopy groups of X for $i < n$ is completely determined by its n-skeleton.

In particular, adding $(n+2)$-cells to X does not affect $\pi_n(X)$ or lower homotopy groups.

3.2.2 Whitehead's Theorem

The following lemma is a generalization of the *compression criterion*:

LEM 3.11 compression lemma

Suppose (Y, B) is a pair of topological spaces such that $B \neq \emptyset$, and (X, A) is a CW pair. Suppose that, for all $n \in \mathbb{Z}_{\geqslant 0}$, whenever $X \setminus A$ has an n-cell, $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$.

Then every map $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$, i.e. can be "compressed" into B.

A map $f : X \to Y$ is called a **weak homotopy equivalence** if it induces an isomorphism on all homotopy groups with respect to all basepoints. Obviously a homotopy equivalence is a weak homotopy equivalence. Whitehead's theorem says that the converse is true for CW complexes:

THM 3.12 Whitehead's theorem

If X and Y are connected CW complexes, then every weak homotopy equivalence $X \to Y$ is also a homotopy equivalence.

3.2.3 CW Approximation

THM 3.13 CW approximation

For any topological space X, there exists a CW complex Z and a weak homotopy equivalence $Z \to X$.

Moreover, such a Z is unique, up to homotopy equivalence.

This Z is called the **CW approximation** of X.

Weak homotopy equivalence is the best we can hope; the *quasi-circle*, obtained by connecting the two ends of a *topologist's sine wave* with a curve, is weakly homotopy equivalent to a point, but not homotopy equivalent to any CW complex.

It turns out that no algebraic topology invariant can distinguish between weakly homotopy equivalent spaces:

PROP 3.14

Weak homotopy equivalences $f : X \to Y$ induces isomorphisms on all homology groups and cohomology groups (and also cohomology rings) for all coefficient groups (rings).

A notable technique involved here is by introducing the **mapping cylinder** M_f, defined for $f : X \to Y$, by attaching one base for the cylinder $X \times [0, 1]$ to Y by f. The noticable feature of this space is that the obvious inclusion map $X \hookrightarrow M_f$ can be homotoped to the map $f : X \to Y$, so in proving such statements we can assume that f is an inclusion.

Let us denote $[X, Y]$ the set of maps $X \to Y$, modulo homotopy rel basepoint. Then the homotopy groups as sets can be written as $\pi_n(X, x_0) = [\mathbb{S}^n, X]$. A weak homotopy equivalence $Y \to Z$ then induces a bijection $[\mathbb{S}^n, Y] \to [\mathbb{S}^n, Z]$. Turns out that this holds not only for spheres but also for all CW complexes:

PROP 3.15 A weak homotopy equivalence $f : Y \to Z$ induces a bijection for every CW complex X:

$$\begin{aligned} f_* : [X, Y] &\to [X, Z], \\ (X \to Y) &\mapsto \left(X \to Y \xrightarrow{f} Z\right). \end{aligned}$$

3.3 Calculating π_n

3.3.1 Excision

Our goal is to calculate the homotopy groups of union of spaces, as in the Mayer-Vietoris sequence. In $n = 1$, we have the nice Seifert-van Kampen theorem. However this fails for $n > 1$. Even the homotopy groups for wedge products can be complex.

As an example, we can prove that $\pi_n(\bigvee_\alpha \mathbb{S}^n) = \bigoplus_\alpha \mathbb{Z}$, whether finite or countably infinite. Consider $\pi_n(\mathbb{S}^n \vee \mathbb{S}^1)$ for $n > 1$. This is isomorphic to π_n of its universal cover, which is $\mathbb{R}$ with a $\mathbb{S}^n$ attached on every point of $\mathbb{Z}$. This space is homotopy equivalent to the wedge sum of countably infinite $\mathbb{S}^n$, so its π_n is the direct sum of countably infinitely many $\mathbb{Z}$. This shows that even finite CW complexes can have infinite homotopy groups.

We do however have the following version of the "excision" property:

THM 3.16 "excision" Let X be a CW complex, A and B are subcomplexes such that $A \cup B = X$. Suppose $A \cap B = C$. If (A, C) is m-connected and (B, C) is n connected, then the inclusion $(A, C) \hookrightarrow (X, B)$ induces an isomorphism on π_i for $i < m + n$ and a surjection for $i = m + n$.

COR 3.17 Freudenthal suspension theorem If X is an $(n-1)$-connected CW complex, then the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n - 1$, and a surjection for $i = 2n - 1$.

This suspension theorem has the following implications. First, suppose that X is $(n-1)$-connected, then SX is n-connected. Second, the maps $\pi_i(X) \to \pi_{i+1}(SX) \to \cdots \to \pi_{i+N}(S^N X) \to \cdots$ is eventually an isomorphism, so this sequence of groups are eventually all isomorphic to a group. This group is called the **stable homotopy group** $\pi_i^s(X)$.

PROP 3.18 If a CW pair (X, A) is r-connected and A is s-connected, where r and s can be zero, then $(X, A) \to (X/A, *)$ induces an isomorphism on π_i for $i \leqslant r + s$ and a surjection for $i = r + s + 1$.

As an application, an **Eilenberg-Mac Lane space** $K(G, n)$ for $n \geqslant 2$ and an abelian group G is a space X such that $\pi_k(X) = G$ if $k = n$ and 0 if $k \neq n$.

THM 3.19 For any abelian group G and any $n \geqslant 2$, there is a unique CW complex, up to homotopy equivalence, that is a $K(G,n)$. This is therefore called *the* $K(G,n)$.

Any group homomorphism between $G = \pi_n(K(G,n))$ can be induced by a continuous map between spaces.

3.3.2 Hurewicz's theorem

The **Hurewicz map** is a map $h : \pi_n(X) \to H_n(X)$ that maps $[f : \mathbb{S}^n \to X]$ to $f_*([\mathbb{S}^n])$. There is also a relative version $h : \pi_n(X,A) \to H_n(X,A)$, that maps $\left[f : \left(\mathbb{D}^n, \mathbb{S}^{n-1}\right) \to (X,A)\right]$ to $f_*([\mathbb{D}^n])$ where $[\mathbb{D}^n]$ is the fundamental class of $(\mathbb{D}^n, \partial\mathbb{D}^n)$.

THM 3.20 Hurewicz's theorem

If a topological space X is $(n-1)$-connected for some $n \geqslant 2$, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$ by h.

COR 3.21 relative Hurewicz's theorems

1. If (X,A) is $(n-1)$-connected for some $n \geqslant 2$ and A is simply connected, then $H_i(X,A) = 0$ for $i < n$ and $\pi_n(X,A) \cong H_n(X,A)$ by h.
2. A map $f : X \to Y$ between simply connected CW complexes is a homotopy equivalence, if it induces isomorphisms on all homology groups.

3.4 Fibrations

3.4.1 Homotopy Lifting and Homotopy Extension

A map $p : E \to B$ is said to have the **homotopy lifting property** (h.l.p.) for a space X, if given any homotopy $g_t : X \times I \to B$ and a map $\tilde{g}_0 : X \to E$ lifting g_0, there exists a homotopy $\tilde{g}_t : X \times I \to E$ lifting g_t.

Dually, a map $i : A \to X$ is said to have th **homotopy extension property** (h.e.p.) for a space Y, if given any homotopy $f_t : A \times I \to Y$ such that f_0 extends to $\tilde{f}_0 : X \to Y$, we can extend f_t to a homotopy $\tilde{f}_t : X \to Y$.

$$\begin{tikzcd}
X \times 0 \arrow[r, "\tilde{g}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "g_t"'] \arrow[ur, dashed, "\exists \tilde{g}_t"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
Y & X \arrow[l, "\tilde{f}_0"'] \arrow[dl, dashed, "\exists \tilde{f}_t"'] \\
(I \to Y) \arrow[u, "t=0"] & A \arrow[l, "f_t"] \arrow[u, "i"']
\end{tikzcd}$$

A **(Hurewicz) fibration** is a map $p : E \to B$ having the h.l.p. for all topological spaces X. A **cofibration** is a map $i : A \to X$ having the h.e.p. for all topological spaces Y.

The h.l.p. for all disks $\mathbb{D}^k$ can imply h.l.p. for all CW complexes. A **Serre fibration** is a map $p : E \to B$ having the h.l.p. for all *CW complexes*, or equivalently, all *disks*.

THM 3.22 If $p : E \to B$ is a Serre fibration, b_0 is the basepoint of B and $x_0 \in F = p^{-1}(b_0)$, then $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all n.

Therefore, if B is path-connected, there is a l.e.s.

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, x_0) \to \pi_{n-1}(F, x_0) \to \cdots$$

Given a fibration $p : E \to B$, denote F_b the fiber at b: $F_b = p^{-1}(b)$. Given a curve $\gamma : I \to B$, view γ as a homotopy $\{*\} \times I \to B$. Then for any $x \in F_{\gamma(0)}$, we can take a lift of γ to $\tilde{\gamma} : I \to E$ such that $\tilde{\gamma}(0) = x$. This gives a map $L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}, x \mapsto \tilde{\gamma}(1) \in F_{\gamma(1)}$.

This L_γ satisfies the following properties:

- If $\gamma \simeq \gamma'$ rel ∂I, then $L_\gamma \simeq L_{\gamma'}$.
- $L_{\gamma'} \circ L_\gamma = L_{\gamma\gamma'}$.

In particular,

PROP 3.23 Given a fibration $p : E \to B$, the fibers F_b are homotopy equivalent for b in the same path-connected component.

Denote $\mathrm{HE}(F)$ by the group of homotopy equivalences $F \to F$ modulo homotopy. If B is path-connected, this L then gives a map $\pi_1(B) \to \mathrm{HE}(F)$, called the **monodromy representation**. For example, if $E \to B$ is a covering map, then L is the deck transformation.

3.4.2 Fiber Bundles

As a special case, fiber bundles are fibrations such that the fibers are *homeomorphic*.

Precisely, a **fiber bundle** is a map $p : E \to B$ such that each point $b \in B$ has a neighbourhood U and a homeomorphism h such that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\text{first coordinate}"] \\
& U &
\end{tikzcd}$$

This h is called a **local trivialization**, because the product $B \times F$ is called a **trivial** bundle on B, and h means that the bundle E is locally trivial.

For any path-connected component U, $p^{-1}(b)$ are all homeomorphic for all $b \in U$. Therefore, if B is path-connected, then for all b in B, $p^{-1}(b) \cong F$. F is called the **fiber** of this fiber bundle.

PROP 3.24 Fiber bundles $E \to B$ are Serre fibrations.

If additionally B is paracompact, then it's also a Hurewicz fibration.

Examples of fiber bundles include: all covering maps, Möbius strip $\to \mathbb{S}^1$, $\mathbb{S}^n \to \mathbb{RP}^n$, etc.

EX 3.25 projective spaces. We have fiber bundles

$$\begin{aligned}
\mathrm{O}(1) \cong \mathbb{S}^0 &\to \mathbb{S}^n && \to \mathbb{RP}^n \\
\mathrm{U}(1) \cong \mathbb{S}^1 &\to \mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1} && \to \mathbb{CP}^n \\
\mathrm{Sp}(2) \cong \mathbb{S}^3 &\to \mathbb{S}^{4n+3} \subseteq \mathbb{H}^{n+1} && \to \mathbb{HP}^n
\end{aligned}$$

where $\mathbb{H}$ is the quaternions (similar reesults hold for $\mathbb{O}$ the octonions, up to $n = 2$). In particular, we have the following **Hopf fibrations** on $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively:

$$\begin{gathered}
\mathbb{S}^0 \to \mathbb{S}^2 \to \mathbb{S}^1, \\
\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2, \\
\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{S}^4, \\
\mathbb{S}^7 \to \mathbb{S}^{15} \to \mathbb{S}^8.
\end{gathered}$$

Also, taking $n = \infty$, we proved that $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}_2, 1)$ and $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

EX 3.26 Lie groups. Let $p : \mathrm{O}(n) \to \mathbb{S}^{n-1}, A \mapsto A \cdot (1, 0, \cdots 0)^\top$. The fiber is $\mathrm{O}(n-1)$. This gives a fiber bundle $\mathrm{O}(n-1) \to \mathrm{O}(n) \to \mathbb{S}^{n-1}$. The l.e.s. of this fiber bundle shows that $\pi_i \mathrm{O}(n) \cong \pi_i \mathrm{O}(n-1)$ if $i \leqslant n-2$. Therefore, for large enough n, $\pi_i \mathrm{O}(n)$ is constant. This constant group is the **stable homotopy group of the orthogonal group** $\pi_i \mathrm{O}$.

There is a *Bott periodicity theorem* that says the stable homotopy groups $\pi_i \mathrm{O}$ are periodic in i:

$i \bmod 8$	1	2	3	4	5	6	7	8
$\pi_i \mathrm{O}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Similar results hold for $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

3.4.3 Homotopy Fiber and the Loop Space

Given fibrations $p_i : E_i \to B$ $(i = 1, 2)$, say a map $f : E_1 \to E_2$ is **fiber-preserving** if $p_1 = p_2 \circ f$, and is a **fiber homotopy equivalence** if it's fiber-preserving, and there is some $g : E_2 \to E_1$ such that $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$ through fiber-preserving maps.

Given a fibration $p : E \to B$ and a map $f : A \to B$, the **pullback fibration** is $f^*E := \{(a,e) \in A \times E : f(a) = p(e)\}$, and $f^*E \to A$ defined by projection is a fibration. f^*E is also denoted $A \times_B E$, the **fiber product**. A pullback is represented in the commutative diagram by a $\lrcorner$ symbol.

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

If $f_0, f_1 : A \to B$ are homotopic, then the pullback fibrations $f_0^*E \to A$ and $f_1^*E \to A$ are fiber homotopy equivalent. Therefore, if B is contractible, then every fibration is fiber homotopy equivalent to the trivial fibration.

Recall the construction of the mapping cylinder: for any map $f : A \to B$, we can construct

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[rr, bend right, "f"'] & M_f \arrow[r, "\simeq"] & B
\end{tikzcd}$$

such that $M_f \to B$ is a homotopy equivalence, and i is a cofibration. We now describe a dual to thie construction, justifying the slogan that "every map is a fibration".

For any map $f : A \to B$, let $E_f = \{(a,\gamma) : a \in A, \gamma : I \to B \text{ a path where } \gamma(0) = f(a)\}$. Then E_f is a subset of $A \times B^I$, where B^I is the set of maps $I \to B$ with the compact-open topology. The fact is that

- $p : E_f \to B, (a,\gamma) \mapsto \gamma(1)$ is a fibration;
- A can be embedded into E_f by taking a to the constant path at a;
- E_f deformation retracts to A by deforming every path γ to its starting point.

Therefore we have a construction

$$\begin{tikzcd}
A \arrow[r, hook, "\simeq"] \arrow[rr, bend right, "f"'] & E_f \arrow[r, "p"] & B
\end{tikzcd}$$

where $A \to E_f$ is a homotopy equivalence, and p is a fibration.

If B is path-connected, the fiber of p is well-defined up to homotopy equivalence. This fiber is called the **homotopy fiber** of the map $f : A \to B$, denoted F_f; explicitly, $F_f = p^{-1}(b) = \{(a,\gamma) : \gamma(0) = f(a), \gamma(1) = b\}$, up to homotopy equivalence.

If $p : E \to B$ itself is a fibration with fiber F, then $E \hookrightarrow E_p$ is a homotopy equivalence, and the homotopy fiber F_p is then homotopy equivalent to the "true" fiber F.

As a special case, take $f : A = \{b_0\} \hookrightarrow B$. Then $E_f = \{\gamma : \gamma(0) = b_0\}$. This is called the **path space** PB of B. The homotopy fiber is $F_f = \{\gamma : \gamma(0) = \gamma(1) = b_0\}$, called the **loop space** ΩB of B. We therefore get a fibration $\Omega B \to PB \to B$. This is the **loop fibration**.

Observe that PB is contractible, by contracting every path to its starting point; therefore, by the l.e.s. of the fibration, $\pi_n(B) \cong \pi_{n-1}(\Omega B)$. In particular, $\Omega K(G,n) \simeq_{\mathrm{w}} K(G, n-1)$, where $\simeq_{\mathrm{w}}$ denotes weak homotopy equivalence. For example, $\Omega\mathbb{C}\mathrm{P}^\infty \simeq_{\mathrm{w}} \mathbb{S}^1$, $\Omega\mathbb{R}\mathrm{P}^\infty \simeq_{\mathrm{w}} \mathbb{S}^0$, and $\Omega\mathbb{H}\mathrm{P}^\infty \simeq_{\mathrm{w}} \mathbb{S}^3$.

A non-trivial fact, proved by Milnor, is that the loop space of a CW complex is always homotopy equivalent to a CW complex.

The important fact about the path space fibration is that it is the *universal* fibration; that is, any fibration $\Omega B \to E' \to B'$ with fiber ΩB is a pullback of the path space fibration $\Omega B \to PB \to B$.

$$\begin{tikzcd}
\Omega B \arrow[r] & E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d] \\
\Omega B \arrow[r] & PK \arrow[r] & K
\end{tikzcd}$$

Given a fibration $F \to E \to B$, we can form a fibration $\Omega B \to F \to E$. This in turn gives a fibration $\Omega E \to \Omega B \to F$, etc. This is called a **fibration sequence**

$$\cdots \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

where every two consecutive maps form a fibration.

3.5 Cohomology via $K(G,n)$

Let $[X, Y]$ denote maps $X \to Y$ modulo homotopy, and $\langle X,Y\rangle$ denote based maps $X \to Y$ modulo homotopy rel basepoints.

The main result will be

THM 3.27 There is a natural bijection

$$T : \langle X, K(G,n)\rangle \xrightarrow{\cong} H^n(X, G)$$

for all $n \geqslant 0$, CW complexes X and abelian groups G.

Let's start by making T more explicit. Let $K = K(G,n)$. Then by universal coefficient theorem and Hurewicz theorem, $H^n(K;G) \cong \mathrm{Hom}(H_n(K;\mathbb{Z}), G) \cong \mathrm{Hom}(\pi_n(K), G) = \mathrm{Hom}(G,G)$. There is a special element in $\mathrm{Hom}(G,G)$, namely the identity map; its pre-image in $H^n(K;G)$, α, is called the **fundamental class** of K. Then T can be written as $T:(f:X\to K)\mapsto f^*\alpha$.

If we forget about basepoints, we then get a map $\langle X,K(G,n)\rangle \to [X,K(G,n)]$. In most cases this won't be a problem; this is a bijection for $n\geqslant 2$, and if G is abelian for $n=1$. For $n=0$ we need to replace H^0 by $\tilde{H}^0$ for based maps.

Since T is a bijection, we would expect that $\langle Y,K(G,n)\rangle$ has a group structure. It turns out that there is a natural group structure when Y is a suspension $Y=SX$, where multiplication is defined via $SX/X\cong SX\vee SX$ as in the case of π_n. If we also take basepoints into consideration, we need to replace suspension by **reduced suspension** $\Sigma X = SX/(x_0\times I)$; therefore, $\langle \Sigma X,K\rangle$ forms a group for based spaces X and K.

For general X which is not a suspension, we need the following adjoint relation

$$\langle \Sigma X,K\rangle \longleftrightarrow \langle X,\Omega K\rangle,$$
$$(f:\Sigma X\to K)\longleftrightarrow (X\to \Omega K, x\mapsto f|_{x\times I}).$$

The group structure on $\langle X,\Omega K\rangle$ is given by composition of loops $\Omega K\times\Omega K\to\Omega K$. For n large enough, $\langle \Sigma^n X,K\rangle\cong\langle X,\Omega^n K\rangle$ is an abelian group.

Inspired by this, an **Ω-spectrum** is a sequence of CW complexes $K_1,K_2,K_3,\cdots$ wich weak homotopy equivalences $K_n\simeq_{\mathrm{w}}\Omega K_{n+1}\simeq_{\mathrm{w}}\Omega^2 K_{n+2}\simeq_{\mathrm{w}}\cdots$.

THM 3.28 If $\{K_n\}$ is an Ω-spectrum, define $h^n(X):=\langle X,K_n\rangle$. This defines a (reduced) cohomology theory.

Therefore, for the Ω-spectrum $\{K_n=K(G,n)\}$, this theorem implies Theorem 3.27.

Theorem 3.27 allows us to prove clains about cohomology by studying the universal example, the fundamental class of $H^n(K,G)$.

EX 3.29 let's use this to prove that, for any CW complex X, the cup product $\alpha\smile\alpha$ for any $\alpha\in H^1(X)$ is zero. We know that $H^1(X;\mathbb{Z})\cong\langle X,K(\mathbb{Z},1)\rangle=\left\langle X,\mathbb{S}^1\right\rangle$. Let u be the fundamental class of $\mathbb{S}^1$, then this isomorphism is expressed as $f^*u\mapsto f$. Therefore, for any $\alpha\in H^1(X)$, there is some $f:X\to\mathbb{S}^1$ such that $\alpha\smile\alpha = f^*u\smile f^*u=f^*(u\smile u)=0$ because $u\smile u=0$ in $H^*\left(\mathbb{S}^1\right)$. We proved the claim for all CW complexes X just by studying $\mathbb{S}^1$!

3.6 Obstruction Theory

Consider the following questions:

Extension problem. Given a CW pair (W,A) and a map $A\to X$, does it extend to $W\to X$?

Lifting problem. Given a fibration $X \to Y$ and a map $W \to Y$, does f lift to a map $W \to X$?

These two questions both generalizes to the

Relative lifting problem. Given a CW pair (W, A), a fibration $X \to Y$, a map $W \to Y$ and a partial lift $A \to X$, is there a global lift $W \to X$ that extends it?

Obstruction theory will tell us that a solution to these problems won't exist if some *obstruction classes* are present.

3.6.1 Postnikov Towers

A **Postnikov tower** for a path-connected space X is a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d] \\
 & X_2 \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & X_1
\end{tikzcd}$$

such that the map $X \to X_n$ induces isomorphisms on all π_i (for $i < n$), and $\pi_i X_n = 0$ for all $i > n$.

We can additionally assume that $X_n \to X_{n-1}$ is a fibration for all n. Let F_n be the homotopy fiber of $X_n \to X_{n-1}$; then by the l.e.s. of a fibration, $F_n = K(\pi_n X, n)$.

We can construct the *inverse limit* of all the spaces X_n, denoted by $\varprojlim_n X_n$. The inverse limit of the maps $X \to X_n$ gives a map $X \to \varprojlim_n X_n$. This map is a weak homotopy equivalence: $X \simeq_{\mathrm{w}} \varprojlim_n X_n$.

A fibration $F \to E \to B$ is called **principal**, if there is a commutative diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\simeq_{\mathrm{w}}"] & E \arrow[r] \arrow[d, "\simeq_{\mathrm{w}}"] & B \arrow[d, "\simeq_{\mathrm{w}}"] & \\
\Omega B \arrow[r] & F' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

where the lower row is a fibration sequence.

THM 3.30 A connected CW complex X has a Postnikov tower such that all maps $X_n \to X_{n-1}$ are principal fibrations, if and only if the actions of $\pi_1 X$ on $\pi_n X$ are trivial for all n.

Consider a portion of the Postnikov tower with principal fibrations:

$$\begin{tikzcd}
 & F_n \arrow[r, "\simeq_w"] \arrow[d] & \Omega B' \arrow[d] & = & K(\pi_n X, n) \\
 & X_n \arrow[r, "\simeq_w"] \arrow[d, bend right] & F' \arrow[d] & & \\
X \arrow[r] \arrow[ur] & X_{n-1} \arrow[r, "\simeq_w"] \arrow[dr, bend right] & E' \arrow[d] & & \\
 & & B' & = & K(\pi_n X, n+1)
\end{tikzcd}$$

Because the fibration $X_n \to X_{n-1}$ is principal, according to the diagram above, we can construct a fibration $X_n \to X_{n-1} \to K(\pi_n X, n+1)$.

A space X is called **simple** if $\pi_1 X$ is abelian, and all actions $\pi_1 X \curvearrowright \pi_n X$ are trivial. For such a space, we can furthur extend the Postnikov tower to have an additional term $X_0 = *$.

3.6.2 Obstruction Classes

Let's focus on the *extension problem* first. Suppose X is simple. Look at its Postnikov tower with principal fibrations. The violet line is the extension we are looking for, and the red lines are the same fibrations as the one the red arrows represents in the previous diagram.

$$\begin{tikzcd}
 & & \vdots \arrow[d] \arrow[dr, bend right] & \\
 & & X_3 \arrow[r] \arrow[d] \arrow[dr, bend right] & K(\pi_4 X, 5) \\
 & & X_2 \arrow[r] \arrow[d] \arrow[dr, bend right, "\text{fibration}"] & K(\pi_3 X, 4) \\
A \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[ur] \arrow[uur] \arrow[dr] & X_1 \arrow[r] \arrow[d] & K(\pi_2 X, 3) \\
W \arrow[rr] \arrow[ur, dashed, "?"] & & X_0 = * &
\end{tikzcd}$$

We already have a trivial map $W \to X_0 = *$. The plan is to induct on n; by induction hypothesis we already have $W \to X_{n-1}$. We with to lift it to $W \to X_n$.

Let $K = K(\pi_n X, n+1)$. Then ΩK is the fiber of the fibration $X_n \to X_{n-1}$. Therefore, X_n is a pullback fibration:

$$\begin{tikzcd}
 & X_{n-1} \times_K PK & \\
 & \wr & \\
\Omega K \arrow[r] & X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n-1} \arrow[d] \\
\Omega K \arrow[r] & PK \arrow[r] & K
\end{tikzcd}$$

The key observation is that, since PK is contractible, a lift $W \to X_n \simeq X_{n-1} \times_K PK$ can be identified with a nullhomotopic map $W \to X_{n-1} \to K$.

$$\begin{tikzcd}
 & & PK \simeq * \arrow[d] \\
W \arrow[r] \arrow[urr] \arrow[rr, dashed, bend right, "\text{nullhomotopic}"'] & X_{n-1} \arrow[r] & K
\end{tikzcd}$$

Now we have a lift on $A \subseteq W$, and a nullhomotopy $A \times I \to K$. All these gives a map $W \cup CA \to K$ where CA is the cone of A with base identified with $A \subseteq W$. Remember that $K = K(\pi_n X, n+1)$; therefore, such a map corresponds to a cohomology class $\omega_n \in H^{n+1}(W \cup CA; \pi_n X)$, which, by excision, is isomorphic to $H^{n+1}(W, A; \pi_n X)$. This ω_n is called the **obstruction class**. A lifting $W \to X_n$ exists, *if and only if* $\omega_n = 0$.

COR 3.31 If X is a path-connected simple CW complex and (W, A) is a CW pair such that $H^{n+1}(W, A; \pi_n X) = 0$ for all n, then the extension problem for (W, A) and X always has a solution.

For the *relative lifting problem* for a CW pair (W, A) and a fibration $F \to X \to Y$, one need to consider a variant of the Postnikov tower.

Suppose $p : X \to Y$ is a fibration with fiber F. The **Moore-Postnikov tower** is a commutative diagram

$$\begin{tikzcd}
& \vdots \arrow[d] & \\
& Z_3 \arrow[d] \arrow[ddr] & \\
& Z_2 \arrow[d] \arrow[dr] & \\
X \arrow[r] \arrow[ur] \arrow[uur] \arrow[rr, bend right, "p"'] & Z_1 \arrow[r] & Y
\end{tikzcd}$$

such that

- every map $X \to Z_n$ induces isomorphisms on π_i for $i < n$ and a surjection on π_n,
- every map $Z_n \to Y$ induces isomorphisms on π_i for $i < n$ and an injection on π_n,
- every map $Z_n \to Z_{n-1}$ is a fibration with fiber $K(\pi_n F, n)$.

Every fibration has a Moore-Postnikov tower, unique up to homotopy equivalence. A Moore-Postnikov tower of *principal* fibrations exists if the action of $\pi_1 X$ on $\pi_n\left(M_p, X\right)$ is trivial for any $n \geqslant 1$. Here M_p is the mapping cylinder of M_p.

Therefore, to solve the relative lifting problem for (W, A) and $p : X \to Y$, we first assume that the action of $\pi_1 X$ on $\pi_n\left(M_p, X\right)$ is trivial for $n \geqslant 1$; we need to additionally assume that the given map $f : W \to Y$ takes $\pi_1 W$ into $p_* \pi_1 X$, so that the induction basis $W \to Z_1$ can be constructed. We also need that $\pi_1 F$ is abelian where F is the fiber of $X \to Y$.

Under all these conditions, we can use a similar argument as before, to the following similar diagram:

$$
\begin{tikzcd}
& & \vdots \arrow[d] & \\
& & Z_3 \arrow[r] \arrow[d] & K(\pi_3F,4) \\
& & Z_2 \arrow[r] \arrow[d, "\text{fibration}"] & K(\pi_2F,3) \\
A \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[uur] \arrow[ur] \arrow[dr] & Z_1 \arrow[r] \arrow[d] & K(\pi_1F,2) \\
W \arrow[rr] \arrow[ur, dashed, "?"'] & & Y &
\end{tikzcd}
$$

In this case the obstruction class ω_n lies in $H^{n+1}(W,A;\pi_nF)$.

Generally, ω_n depends on the choice of the lift $W \to Z_n$. Therefore, even if a non-zero obstruction exists, it may not imply that a solution doesn't exist; it may be due to an inappropriate choice of previous lifts.

However, the converse is true; if all obstruction classes are zero, then a solution do exist.

There is, however, a case where the choice of ω_n is canonical. If F is $(n-1)$-connected, then $H^{n+1}(W,A;\pi_nF)$ is the first possible non-zero cohomology; the obstruction class $\omega_n \in H^{n+1}(W,A;\pi_nF)$ is then called the **first** or the **primary obstruction**, and does not depend on the choices of lifts in the previous steps.

EX 3.32 suppose $F \to E \xrightarrow{p} B$ is a fibration and B is a CW complex, and suppose that F is contractible. Consider the relative lifting problem for CW pair (B, B^0), fibration $p : E \to B$ and map $\mathrm{id} : B \to B$. A fact is that the Moore-Postnikov tower of principal fibrations always exists for such a fibration. Also, $p_*\pi_1E = \pi_1B$ because $\pi_1F = 0$. Therefore, obstruction theory applies; since F is contractible, all obstruction classes are zero, so a lift $B \to E$ exists. This lift is a section $B \to E$.

Therefore, if $F \to E \to B$ is a fibration with F contractible, then there is a section $B \to E$.

EX 3.33 there are various applications of the previous example.

1. We prove that every closed n-dimensional manifold M has a Riemannian metric. Let E be all positive definite bilinear maps $\mathrm{T}M \times \mathrm{T}M \to \mathrm{T}M$. A Riemannian metric is then a section $M \to E$. $E \to M$ is a fiber bundle, with fiber the set of positive definite bilinear forms on $\mathbb{R}^n$. This is a convex subspace of $\mathbb{R}^{n^2}$, so it's contractible. Therefore, a section does exist.

2. When does a closed orientable n-dimensional manifold M has a nowhere vanishing vector field?

   Let $\mathrm{UT}_pM = \{v \in \mathrm{T}_pM : \|v\| = 1\}$. Then a nowhere vanishing vector field is a section $M \to \mathrm{UT}M$. The fiber bundle $\mathrm{UT}M \to M$ has fiber $\mathrm{UT}_pM \cong \mathbb{S}^{n-1}$, so the obstruction class ω_m lies in $H^{m+1}(M, \pi_m\mathbb{S}^{n-1})$. This is zero for $m < n-1$. Since M is n-dimensional, it's also zero for $m > n-1$. So the only possible obstruction is $\omega_{n-1} \in H^n(M;\mathbb{Z}) \cong \mathbb{Z}$.

   A fact is that the Poincaré pairing $\langle \omega_{n-1}, [M] \rangle$ is equal to $\chi(M)$. Therefore, a nowhere vanishing vector field exists if and only if $\chi(M) = 0$.

EX 3.34 a particularly important application is the *principal bundle*.

If $F \to E \to B$ is a fiber bundle, G is a topological group, G acts on E preserving fibers, such that G act on every fiber F_b freely and transitively (so that $G \cong F$ by orbit-stabilizer), and $E/G = B$, say this fiber bundle a **principal G-bundle**. Examples include: if H is a closed Lie subgroup of G then $H \to G \to G/H$ is a principal H-bundle.

A theorem by Milnor says that, for any topological group G, there is a principal G-bundle $G \to EG \to BG$ such that EG is contractible. This BG is called the **classifying space** of G.

This principal bundle is *universal*, in the sense that every other principal G-bundle is a pullback of it. To see this, let $E \times_G EG$ be the quotient of $E \times EG$ by action of G. Then we have two fiber bundles

$$EG \to E \times_G EG \to B$$
$$E \to E \times_G EG \to BG$$

Since EG is contractible, there is a section $B \to E \times_G EG$, which gives a map $B \to E \times_G EG \to BG$. This map then gives a principal fibration

$$G \to E \to B \to BG,$$

showing that $G \cong \Omega BG$. This is actually the origin of the notion of *principal fibration*.

3.7 Cohomology of Fiber Bundles

The cohomology of fiber bundles is generally hard to compute. Even for the trivial bundle $E = F \times B$, we already need the Künneth formula. The general computation requires the notion of Serre spectral sequences. However, there is an important special case where spectral sequences are not required:

THM 3.35
Leray-Hirsch

Suppose R is a commutative ring, $F \xrightarrow{i} E \xrightarrow{p} B$ is a fiber bundle, such that

1. $H^n(F;R)$ is a finitely generated free R-module for all n;
2. $i^* : H^*(E;R) \to H^*(F;R)$ is surjective.

Let $c_j \in H^*(E;R)$ $(j \in J)$ such that $\left\{i^* c_j\right\}$ forms a basis of $H^*(F;R)$.

Then the map

$$\begin{aligned} \Phi : H^*(B;R) \otimes_R H^*(F;R) &\to H^*(E;R), \\ b \otimes i^* c_j &\mapsto p^* b \smile c_j \end{aligned}$$

is an isomorphism of $H^*(B;R)$-modules. Actually, $H^*(E;R)$ is a free $H^*(B;R)$-module generated by $\left\{c_j\right\}$.