Representation Theory

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Section 0
Introduction

Representation theory began with an observation by Dedekind. For a finite abelian group G and formal variables x_g for $g \in G$, define a $|G| \times |G|$ matrix X_G by defining the entry at row g column h to be $x_{g \cdot h}$. Dedekind computed its determinant $\det X_G$ for various abelian groups and observed that it can always be factored into linear factors.

He then wrote a letter to Frobenius to ask for a proof. Frobenius then studied this behaviour for abelian and non-abelian groups, and gave a proof as well as a generalization to non-abelian groups:

THM 1

- Suppose G is a finite abelian group. Then

$$\det X_G = \prod_{\chi : G \to \mathbb{C}^\times} \left(\sum_{g \in G} \chi(g) x_g \right), \tag{1}$$

  where the product is over all group homomorphisms $\chi : G \to \mathbb{C}^\times$. Such a homomorphism is called a **group character**.

- Suppose G is a finite group, not necessarily abelian. Then

$$\det X_G = \prod_{j=1}^{r} p_j(\underline{x})^{\deg p_j}, \tag{2}$$

  where p_j are pairwise non-proportional irreducible polynomials and r is the number of conjugacy classes of G.

This marks the beginning of representation theory.

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Section 1

Basic Notions of Representation Theory

Throughout this section k will denote a general field as the base field.

1.1 Representation of Algebras

1.1.1 Associative Algebras

An **(associative) algebra** over k is a vector space over k equipped with a bilinear associative multiplication *with unit*. An associative algebra is called **commutative** if the multiplication is commutative. An **(algebra) homomorphism** is a linear map between algebras that preserves multiplication and unit.

EX 1.1

- If V is a vector space, then $A = \mathrm{GL}(V)$ is an algebra, where multiplication is composition.
- $A = k\langle x_1, \cdots, x_n\rangle$, the **free algebra** over k, is generated by $x_1, \cdots, x_n$ with no relations.
- For a group G, the **group algebra** $A = k[G]$ has basis $\left\{a_g : g \in G\right\}$ and multiplication defined by $a_g \cdot a_h = a_{g \cdot h}$.

A **left** / **right ideal** of an algebra is a vector subspace $I \subseteq A$ such that for every $a \in A$, $aI \subseteq I$ / $Ia \subseteq I$. A (two-sided) ideal is a subspace that is both a left ideal and a right ideal. If I is an ideal, we can form the quotient algebra $A/I = \{a + I : a \in A\}$, with the quotient (algebra) homomorphism $\pi : A \to A/I, a \mapsto a + I$. The ideal generated by a subset S is denoted $\langle S\rangle$.

An algebra can be generated by generators and relations. Let $k\langle x_1, \cdots, x_n\rangle$ be the free algebra, and $f_1, \cdots, f_m \in k\langle x_1, \cdots, x_n\rangle$ be the relations; then $A = k\langle x_1, \cdots, x_n\rangle / \langle f_1, \cdots, f_m\rangle$ is called the algebra generated by $x_1, \cdots, x_n$ subject to the relations $f_1, \cdots, f_m$.

1.1.2 Representations

DEF 1.2 A **representation** of an associative algebra A, also called a **left A-module**, is a vector space V with an algebra homomorphism $\rho_V : A \to \mathrm{End}(V)$. We denote $\rho_V(a)(v)$ by $a \cdot v$.

EX 1.3

- $V = A$ itself can be made into a representation, by defining $\rho_V(a)$ to be the left multiplication by a. This is called the **regular representation**.
- If $A = k\langle x_1, \cdots, x_n\rangle$, then a representation of A is equivalent to a vector space V with n linear operators (in correspondence to the variables x_i).

A representation is called **faithful** if ρ_V is injective.

A **subrepresentation** of V is a subspace $W \subseteq V$, such that $\rho_V(W) \subseteq W$. A representation is called **irreducible** or **simple** if its only subrepresentations is 0 and itself.

If V_1 and V_2 are two representations, a **(representation) homomorphism** $V_1 \to V_2$ is a linear map $\varphi : V_1 \to V_2$ such that $\varphi(a \cdot v) = a \cdot \varphi(v)$ for all $a \in A$ and $v \in V_1$. φ is called a **isomorphism** if it's an isomorphism of vector spaces. In this case, V_1 is **isomorphic** to V_2.

If V_1 and V_2 are two representations, we can make their direct sum $V_1 \oplus V_2$ into a representation by $a \cdot (v_1 \oplus v_2) = (a \cdot v_1) \oplus (a \cdot v_2)$. This is called the **direct sum** of representations. A representation V is called **indecomposable** if it's not isomorphic to a direct sum.

By definition irreducible representations are indecomposable; the converse is in general not true. The basic questions of representation theory is then the **classification of irreducible and indecomposable representations**.

PROP 1.4 Suppose V_1 and V_2 are two representations of A, φ is a non-zero representation homomorphism $V_1 \to V_2$.

- If V_1 is irreducible, then φ is injective.
- If V_2 is irreducible, then φ is surjective.
- If V_1 and V_2 are both irreducible, then φ is an isomorphism.

COR 1.5 If k is additionally *algebraically closed* and V is a finite dimensional irreducible representation of A, then all representation homomorphisms $V \to V$ are scalar multiplications.

If k is algebraically closed and A is additionally commutative, then every irreducible representation of A is 1-dimensional.

EX 1.6

- $A = k\langle x \rangle$ where k is algebraically closed. Since A is commutative, irreducible representations of A are $V = k$ and $\rho_V(x) = \lambda$ for some $\lambda \in k$.

  A representation of A is a vector space with a linear operator; therefore, indecomposable representations of A are linear operators that has only one Jordan block.

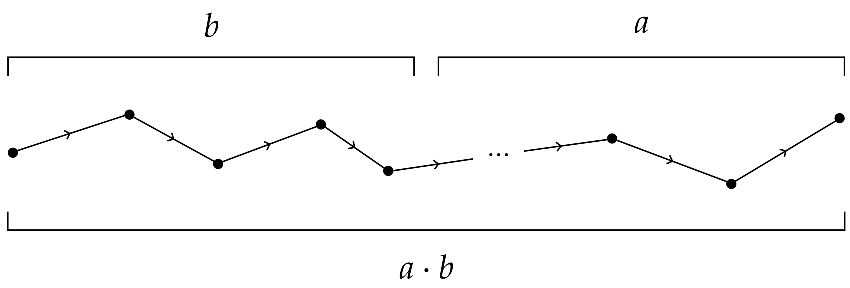
- $A = k[G]$ for some group G. A representation of A is a vector space V with a homomorphism $G \to \mathrm{GL}(V)$; such a homomorphism is called a **representation** of the group G.

1.2 Representation of Quivers

A **quiver** is a directed graph, possibly with self-loops and parallel edges. For an edge h pointing from h' to h'', h' is called the **source** and h'' is called the **target**. The source of h will always be denoted h', and the target will always be denoted h''.

A **representation** of a quiver Q is an assignment to every vertex g a vector space V_k and to every edge $h' \to h''$ a linear map $V_{h'} \to V_{h''}$.

For a quiver Q with finitely many vertices, we can define its **path algebra** P_Q, generated by the set of paths[1] in Q, and multiplication given by path concatenation, zero if two paths cannot be concatenated.



At each vertex i, there is a trivial path p_i, consisting of only one vertex, i itself. If Q has finitely many vertices, then the unit of the path algebra P_Q is $1 = \sum_{\text{vertex } i} p_i$.

PROP 1.7 A representation of a quiver Q is the same as a representation of its path algebra P_Q.

[1]actually, *walks*, as we allow repeated vertices and edges; we will still use the word *paths* however.

1.3 Representation of Lie Algebras

For introductory materials on Lie algebras and their representations, we refer to the Lie Groups and Lie Algebras course notes.

A **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

EX 1.8 adjoint representation

If $V = \mathfrak{g}$, then $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, $x \mapsto [x, -]$ is a Lie algebra representation, called the **adjoint representation**. This ρ is usually denoted ad, and $\mathrm{ad}_x(y) = [x, y]$.

Let $\mathfrak{g}$ be a Lie algebra, with basis $\{x_1, \cdots, x_n\}$. Suppose $\left[x_i, x_j\right] = \sum c_{ij}^k x_k$, where $c_{ij}^k \in k$ are called the **structure constants**. The **universal enveloping algebra**, $\mathcal{U}(g)$, is the associative algebra generated by $x_1, \cdots, x_n$ subject to relations $x_i x_j - x_j x_i = \sum c_{ij}^k x_k$. Representations of a Lie algebra are the same as representations of its universal enveloping algebra.

THM 1.9 The ordered monomials $x_1^{d_1} \cdots x_n^{d_n}$ with $d_i \in \mathbb{Z}_{\geqslant 0}$ form a basis of $\mathcal{U}(g)$.

If V and W are representations of $\mathfrak{g}$, then we can form a Lie algebra representation on $V \otimes W$, the **tensor representation**, by

$$\begin{aligned} \rho_{V\otimes W} : \mathfrak{g} &\to \mathfrak{gl}(V \otimes W), \\ x &\mapsto \rho_V(x) \otimes \mathrm{id} - \mathrm{id} \otimes \rho_W(x), \end{aligned} \tag{3}$$

and a representation on the dual space V^*, the **dual representation**, by

$$\begin{aligned} \rho_{V^*}(x) : \mathfrak{g} &\to \mathfrak{gl}(V^*), \\ x &\mapsto -\rho_V(x)^\top, \end{aligned} \tag{4}$$

$\top$ denoting adjoint. The form of these representations are deduced from the tensor and dual representation of its corresponding Lie group representation.

1.4 Example: Representations of $\mathfrak{sl}_2$

Suppose $\mathfrak{sl}_2$ is the special linear Lie algebra over $\mathbb{C}$. $\mathfrak{sl}_2$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \; f = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}, \tag{5}$$

with relations

$$[h, e] = 2e, \; [h, f] = -2f, \; [e, f] = h. \tag{6}$$

Therefore, a representation of $\mathfrak{sl}_2$ is the same as a vector space V over $\mathbb{C}$ with three linear operators E, F, H, satisfying $HE - EH = 2E$, $HF - FH = -2F$, $EF - FE = H$. We now discuss its finite dimensional irreducible representations.

Step 1. Pick an eigenvalue of H with largest real part, and denote $\overline{V}(\lambda)$ its generalized eigenspace, $\overline{V}(\lambda) = \{v \in V : (H - \lambda I)^n v = 0 \text{ for some } n \geqslant 0\}$. Since $(H - (\lambda+2)I)E = E(H - \lambda I)$, we have $(H - (\lambda+2)I)^n E = E(H - \lambda I)^n$; therefore, for every $v \in \overline{V}(\lambda)$, $Ev \in \overline{V}(\lambda + 2)$. But λ has the largest real part, so $E|_{\overline{V}(\lambda)} = 0$.

Step 2. Let $P_n(x) = n! x(x-1)\cdots(x-n+1)$. By induction, if $w \in V$ such that $Ew = 0$, then for any $n > 0$, $E^n F^n w = P_n(H) w$.

Step 3. If $v \in \overline{V}(\lambda)$, then by a similar argument as Step 1, $Fv = \overline{V}(\lambda - 2)$, so $F^n v \in \overline{V}(\lambda - 2n)$. Since $\dim V < \infty$, H has only finitely many eigenvalues; therefore, for some N, $\overline{V}(\lambda - 2N) = 0$, so $F^N v = 0$ for $v \in \overline{V}(\lambda)$.

Step 4. By Step 2, $E^N F^N|_{\overline{V}(\lambda)} = 0 = P_N(H)|_{\overline{V}(\lambda)}$. As P_N has no multiple roots, this tells us that H is diagonalizable on $\overline{V}(\lambda)$; therefore, $\overline{V}(\lambda)$ is the eigenspace of λ, $V(\lambda) = \{v \in V : Hv = \lambda v\}$.

Step 5. Suppose $v \in V(\lambda)$. Let N be the smallest integer such that $F^N v = 0$. By induction $EF^k v = kF^{k-1}Hv - (k-1)kF^{k-1}v = k(\lambda - k + 1)F^{k-1}v$; let $k = N$, then $EF^N v = 0$ but $F^{N-1}v \neq 0$. Therefore, $N(\lambda - N + 1) = 0$, so $\lambda = N - 1$.

Step 6. Therefore, for any positive integer N, there is an irreducible representation of $\mathfrak{sl}_2$ with basis

$$v_0 = v,\ v_1 = Fv,\ v_2 = \frac{F^2 v}{2!},\ \cdots,\ v_{N-1} = \frac{F^{N-1}v}{(N-1)!}, \tag{7}$$

with actions

$$\begin{tikzcd}
 & \mathbb{C}v_0 \arrow[r, shift left, "F"] \arrow[loop below, "H"] & \mathbb{C}v_1 \arrow[l, shift left, "E"] \arrow[r, shift left, "F"] \arrow[loop below, "H"] & \mathbb{C}v_2 \arrow[l, shift left, "E"] \arrow[r, shift left, "F"] \arrow[loop below, "H"] & \cdots \arrow[l, shift left, "E"] \arrow[r, shift left, "F"] & \mathbb{C}v_{N-1} \arrow[l, shift left, "E"] \arrow[loop below, "H"] \\
\text{with eigenvalue} & N-1 & N-3 & N-5 & \cdots & 1-N
\end{tikzcd}$$

and this representation is denoted V_λ where $\lambda = N - 1$; moreover they are all irreducible representations of $\mathfrak{sl}_2$.

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Section 2
General Results on Representation Theory

From here on we will assume that the base field k is an algebraically closed field.

2.1 Semisimple Representations

DEF 2.1 A representation V of A is called **semisimple** if it is the direct sum of some irreducible representations.

EX 2.2 If V is an irreducible representation of A and $\rho_V : A \to \mathrm{End}(V)$, then the vector space $\mathrm{End}(V)$ can be made into a representation of A by $a \cdot A = \rho_v(a)A$ for $A \in \mathrm{End}(V)$. $\mathrm{End}(V)$ is (representation) isomorphic to $V^{\oplus n}$ where $n = \dim V$, so $\mathrm{End}(V)$ is a semisimple representation of A.

Let us denote $V^{\oplus n}$ by nV. Every semisimple representation of A has the form $V = \bigoplus_{i=1}^{n} n_i V_i$, where V_i are pairwise non-isomorphic finite dimensional irreducible representations of A. If W_i is a vector space of dimension n_i, then $n_i V_i \cong V_i \otimes W_i$, where the action of A on W_i is trivial (W_i only records the multiplicity of V_i). By Schur's lemma, W_i is isomorphic as a vector space to $\mathrm{Hom}_A(V_i, V)$; therefore,

$$V = \bigoplus_{i=1}^{n} V_i \otimes \mathrm{Hom}_A(V_i, V). \tag{8}$$

This is called the **isotypical decomposition**.

PROP 2.3 Suppose V_i $(1 \leqslant i \leqslant m)$ are pairwise non-isomorphic finite dimensional irreducible representations of A, and $V = \bigoplus_{i=1}^{m} n_i V_i$. Suppose W is a subrepresentation of V. Then $W \cong \bigoplus_{i=1}^m r_i V_i$ where $r_i \leqslant n_i$, and the inclusion $\varphi : W \hookrightarrow V$ is of the form $\varphi = \bigoplus_{i=1}^m \varphi_i$, where $\varphi_i : r_i V_i \to n_i V_i$, $\left(v_1, \cdots, v_{r_i}\right) \mapsto \left(v_1, \cdots, v_{r_i}\right) \cdot X_i$ and X_i is a $r_i \times n_i$ matrix of full rank.

COR 2.4 If V is a finite dimensional representation of A, $v_1, \cdots, v_n$ are linearly independent vectors in V. Then for any $w_1, \cdots, w_n \in V$, there is some $a \in A$ such that $av_i = w_i$ for all i.

COR 2.5
1. Suppose $V = \bigoplus_{i=1}^m V_i$ where (V_i, ρ_i) are pairwise non-isomorphic finite dimensional irreducible representations of A.

   Then $\bigoplus_{i=1}^m \rho_i : A \to \bigoplus_{i=1}^m \operatorname{End}(V_i)$ is surjective.
2. Suppose V is a finite dimensional representation of A.

   Then V is irreducible if and only if $\rho : A \to \operatorname{End}(V)$ is surjective.

2.2 Semisimple Algebras

DEF 2.6 An associative algebra A is called **semisimple** if every finite dimensional representation of it is semisimple.

We will classify all semisimple algebras in this section. We first look at an important class of semisimple algebras, the *matrix algebras*.

2.2.1 The Matrix Algebra

Let d_i $(1 \leqslant i \leqslant r)$ be distinct integers, $V_i = k^{d_i}$ and $M_i = \operatorname{End}(V_i) = \operatorname{Mat}_{d_i \times d_i}(k)$. Let $A = \bigoplus_{i=1}^r M_i$. We will discuss the representations of A.

Obviously, each V_i itself is a representation of A by the projection $\mathrm{pr}_i : A \to M_i = \operatorname{End}(V_i)$, and this representation is irreducible since A contains all operators on V_i.

Suppose now that X is an n-dimensional representation of A. We can form the **dual representation** on the dual space X^*; however this representation is for the **opposite algebra** A^{op}, the same algebra but with the order of multiplication reversed. This representation is given by $(a \cdot f)(v) = f(a \cdot v)$, for $a \in A, f \in X^*$ and $v \in X$. In the particular case of a matrix algebra, we actually have $A^{\mathrm{op}} \cong A$, by the isomorphism $B \mapsto B^\top$; therefore, X^* is a representation of A.

Pick a basis $\{y_1, \cdots, y_n\}$ of X^* and let $\varphi : A^{\oplus n} \to X^*$, $(a_1, \cdots, a_n) \mapsto \sum_{i=1}^n a_i y_i$. Since A contains all scalar multiplications, φ is surjective. Therefore, φ^* gives a injection $X \to (A^{\oplus n})^*$.

Through the map $\operatorname{Mat}_{d \times d}(k) \to \operatorname{Mat}_{d \times d}(k)^*$, $B \mapsto \operatorname{tr}(B^\top \cdot -)$, $(A^{\oplus n})^*$ is isomorphic to $A^{\oplus n}$ *as representations of* A. Therefore,

$$\varphi^* : X \to (A^{\oplus n})^* \cong A^{\oplus n} = \left(\bigoplus_{i=1}^{r} \operatorname{End}(V_i)\right)^{\oplus n} \cong \left(\bigoplus_{i=1}^{r} d_i V_i\right)^{\oplus n}. \tag{9}$$

Therefore, there is an injection $X \to \bigoplus_{i=1}^{r} n d_i V_i$; by Proposition 2.3, X is the direct sum of some V_i's.

In corollary, all irreducible representations of A are V_i, and all representations of A are direct sums of V_i.

2.2.2 Classification of Semisimple Algebras

In this section, we will see that every semisimple algebra is isomorphic to a matrix algebra. Also, we will justify the name "*semisimple*": a semisimple algebra is equivalent to a direct sum of simple algebras (ones with no non-trivial ideals). We will use the concept of *filtrations*; a more detailed examination of filtrations will be postponed.

Suppose V is a finite dimensional representation of A. A **finite filtration** of V is a sequence of subrepresentations

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V. \tag{10}$$

Any finite dimensional representation V admits a finite filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$, such that each quotient V_i / V_{i-1} is irreducible.

DEF 2.7 The **radical** of an algebra A, denoted $\operatorname{rad} A$, is the set of elements $a \in A$ that, for all irreducible representation (V, ρ) of A, $\rho(a) = 0$.

PROP 2.8 The radical of A is a two-sided ideal. It is the largest nilpotent ideal in A; every nilpotent ideal is contained in $\operatorname{rad} A$.

THM 2.9 Any finite dimensional algebra A has only a finite number o finite dimensional irreducible representations up to isomorphism. If we list them as $V_1, \cdots, V_r$, then

$$A / \operatorname{rad} A \cong \bigoplus_{i=1}^{r} \operatorname{End}(V_i). \tag{11}$$

COR 2.10 For any finite dimensional algebra A,

$$\sum_{\substack{V \text{ finite dimensional} \\ \text{irreducible representation}}} (\dim V)^2 \leqslant \dim A. \tag{12}$$

EX 2.11 Let A be the algebra of all upper-triangular matrices. For $1 \leqslant i \leqslant n$ let $(V_i = k, \rho_i)$ be the representation given by $\rho_i(X) = X_{ii}$. They are clearly irreducible and pairwise non-isomorphic. Let B be the subalgebra of strictly upper-triangular matrices. Then B is nilpotent and $\operatorname{rad} A \supseteq B$. But $A / \operatorname{rad} A$ is n dimensional and $\sum_{i=1}^{n} (\dim V_i)^2 = n$, it turns out that $\operatorname{rad} A = B$ and V_i are all irreducible representations.

PROP 2.12

- Finite dimensional simple algebras are exactly those isomorphic to $\mathrm{Mat}_{d\times d}(k)$ for some d.
- For finite dimensional algebras, the following are equivalent:
  1. $\operatorname{rad} A = 0$.
  2. $\sum_{V \text{ finite dimensional irreducible}} (\dim V)^2 = \dim A$.
  3. Any finite dimensional representation of A is semisimple, i.e. A semisimple.
  4. The *regular* representation of A is semisimple.
  5. $A \cong \bigoplus_{i=1}^r \mathrm{Mat}_{d_i\times d_i}(k)$ for some d_i; i.e., A is the direct sum of some simple algebras.

2.3 Characters

Throughout this section A will be an *arbitrary* associative algebra.

DEF 2.13

Suppose (V,ρ) is a representation of A. The **character** of it is

$$\begin{aligned} \chi_V : A &\to k, \\ a &\mapsto \operatorname{tr} \rho(a). \end{aligned} \tag{13}$$

If V is irreducible, call χ **irreducible**.

If we let $[A,A]$ be the commutator subalgebra, the subalgebra generated by elements of the form $xy-yx$, then characters actually factors through $A/[A,A]$.

THM 2.14

1. Characters of distinct irreducible representations of A are linearly independent.
2. If A is *finite dimensional* and semisimple, then all irreducible characters of A are distinct, and they form a basis for $(A/[A,A])^*$.

2.4 Jordan-Hölder Theorem and Krull-Schmidt Theorem

THM 2.15

Suppose that V is a finite dimensional representation of A. Then V admits a finite filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ such that each V_i/V_{i-1} is irreducible.

Moreover, every two such filtrations are equal in length, and the quotients V_i/V_{i-1} are isomorphic up to a permutation. The number n is called the **length** of V, and the quotients are called the **Jordan-Hölder series** of V.

LEM 2.16

Suppose W is indecomposable and $\theta, \theta' \in \mathrm{End}_A(W)$.

1. θ is either an isomorphism or nilpotent.
2. If θ and θ' are both nilpotent, then so is $\theta + \theta'$.

THM 2.17 Any finite dimensional representation of an algebra A can be uniquely (up to isomorphisms and a permutation) decomposed into a direct sum of indecomposable representations.

2.5 Representations of Tensor Products

If A and B are algebras, then their tensor product $A \otimes B$ is also an algebra, with multiplication $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$. If V and W are representations of A and B respectively, then $V \otimes W$ is also a representation with componentwise action. Recall that $\operatorname{End}(V \otimes W) = \operatorname{End}(V) \otimes \operatorname{End}(W)$.

THM 2.18

1. If V and W are irreducible, then $V \otimes W$ is an irreducible representation of $A \otimes B$.
2. Any finite dimensional representation of $A \otimes B$ can be uniquely expressed in the form $V \otimes W$.

Proof.

1. The density theorem says that $A \to \operatorname{End}(V)$ and $B \to \operatorname{End}(W)$ are surjective. Therefore, $A \otimes B \to \operatorname{End}(V \otimes W)$ is also surjective, hence the irreducibility.
2. By taking $A' = \operatorname{im} \rho_V$ and $B' = \operatorname{im} \rho_W$, we may assume that $A = A'$ and $B = B'$ are finite dimensional.

   Claim. $\operatorname{rad}(A \otimes B) = \operatorname{rad} A \otimes B + A \otimes \operatorname{rad} B$.

   If the Claim is true, then

   $$\begin{aligned} A \otimes B / \operatorname{rad}(A \otimes B) &= A/\operatorname{rad} A \otimes B/\operatorname{rad} B \\ &= \Big(\bigoplus_{V \text{ fin. diml. irr.}} \operatorname{End}(V) \Big) \otimes \Big(\bigoplus_{W \text{ fin. diml. irr}} \operatorname{End}(W) \Big) \\ &= \bigoplus_{V,W} \operatorname{End}(V \otimes W), \end{aligned} \tag{14}$$

   therefore proving the result.

   Prove of the claim: let $I = \operatorname{rad} A \otimes B + A \otimes \operatorname{rad} B$. I is nilpotent, and $A \otimes B/I = \bigoplus_{V,W} \operatorname{End}(V \otimes W)$ by the above calculation is semisimple. Therefore, I is the largest nilpotent ideal, hence the radical.

□

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Section 3 Representations of Finite Groups

Let G be a *finite* group, k algebraically closed and $k[G]$ the group algebra.

THM 3.1 $k[G]$ is semisimple if and only if $|G| \in k$ is not zero (or equivalently $\operatorname{char} k > 0$ and $\operatorname{char} k \nmid |G|$).

Therefore, our general results of semisimple algebras apply. In particular, $k[G] = \bigoplus_{V \text{ irr}} \operatorname{End}(V)$, so

$$|G| = \sum_{V \text{ irr}} (\dim V)^2. \tag{15}$$

This is called the **sum of squares formula**.

3.1 Characters

Define the vector space of **class functions**

$$F_c(G,k) = \{f : G \to k : \forall g,h, f(ghg^{-1}) = f(h)\}. \tag{16}$$

Suppose G is a *finite* group and $|G| \neq 0$ in k. Recall that the character χ_V of a representation (V, ρ_V) is defined as $\chi_V(g) = \operatorname{tr} \rho_V(g)$.

THM 3.2 Characters of irreducible representations form a basis of $F_c(G,k)$.

In particular, the number of irreducible representations of G is equal to the conjugacy classes of G.

COR 3.3 If $\operatorname{char} k = 0$, then two representations of G are isomorphic if and only if their character are equal.

We are particularly interested in the case $k = \mathbb{C}$. We can define a G-invariant Hermitian inner product on $F_c(G, \mathbb{C})$ by $(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$.

THM 3.4 If V and W are representations, then $(\chi_V, \chi_W) = \dim \operatorname{Hom}_G(V, W)$.

In particular, the characters of irreducible representations are orthogonal.

Proof. Let $p = \frac{1}{|G|} \sum_{g \in G} g \in k[G]$ be the **symmetrizer.** $\rho_V(p)$ is a projection from V to the stabilizer subspace V^G, so $\chi_U(p) = \dim V^G$.

Therefore,

$$\begin{aligned}(\chi_V, \chi_W) &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{V \otimes W^*}(g) \\ &= \operatorname{tr} \rho_{V \otimes W^*}(p) \\ &= \dim(V \otimes W^*)^G \\ &= \dim \operatorname{Hom}_G(W, V).\end{aligned} \tag{17}$$

□

Therefore, by the isotypical decomposition, the *regular* representation of G can be decomposed as $\bigoplus_{V \text{ irr}} V^{\oplus(\chi_{\text{regular}}, \chi_V)}$.

3.2 Complex Representations

A *finite dimensional* representation V over $\mathbb{C}$ of a group G is called **unitary** if there exists a G-invariant positive definition Hermitian form on V.

By taking the symmetrizer, any *finite dimensional* representation of a *finite* group G has a unitary structure. If V is moreover irreducible, then this structure is unique up to scaling by a positive real.

As a corollary, by taking orthogonal complements, every finite dimensional representation of a *finite* group G is completely reducible.

Let V over $\mathbb{C}$ be a irreducible representation of G and $\{v_1, \cdots, v_n\}$ an orthonormal basis of V with respect to the unitary structure. Let $t_{ij}^V(g) = \left(\rho_V(g)v_i, v_j\right)$.

PROP 3.5 If V and W are both irreducible representations of G, then

$$\left(t_{ij}^V, t_{kl}^W\right) = \begin{cases} \frac{1}{\dim V} \text{ if } V \cong W \text{ and } i = k, j = l \\ 0 \text{ otherwise.} \end{cases} \tag{18}$$

In particular $\left\{t_{ij}^V\right\}$ form an orthogonal basis for $F(G, \mathbb{C}) = \{f : G \to \mathbb{C}\}$.

3.2.1 The Character Table

Since the number of irreducible representations of a *finite* group is equal to the number of conjugacy classes, we can draw a table with columns the conjugacy classes, and rows the irreducible representations. On column C row V, we fill in the character $\chi_V(C)$ (conjugate elements share the same character). This forms a square matrix on $\mathbb{C}$, called the **character table** of G.

There are several properties of this square matrix.

1. The conjugacy class of e is e itself, and $\chi_V(e) = \dim V$. Therefore, column e are the dimensions of irreducible representations. By the sum of squares formula, the sum of squares of the first column is equal to $|G|$.
2. Irreducible characters are orthogonal. Therefore, for any two different rows V_1, V_2, $\sum_C |C| \cdot \chi_{V_1}(C)\overline{\chi_{V_2}(C)} = 0$. For any row V, $\sum_C |C| \cdot |\chi_V(C)|^2 = |G|$.
3. Therefore, by properties of a orthogonal square matrix, the *columns* of a character table are also orthogonal.

EX 3.6 Character tables of $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{A}_4$.

The character table of $\mathfrak{S}_3$ can be constructed from the knowledge of

- a trivial representation,
- a 1-dimensional sign representation,
- sum of squares formula, and
- orthogonality of columns.

χ	e $\# = 1$	(12) $\# = 3$	(123) $\# = 2$
trivial	1	1	1
sign	1	−1	1
$(\mathbb{C}^2)$	2	0	−1

The 2-dimensional representation can be generalized constructed as follows. $\mathfrak{S}_n$ acts by permutation of basis on $\mathbb{C}^n$, and there is an obvious invariant subspace $v_1 + \cdots + v_n = 0$. $\mathbb{C}^n/(v_1 + \cdots + v + n = 0)$ is an irreducible representation of $\mathfrak{S}_n$.

The character table of $\mathfrak{S}_4$ can be constructed from the knowledge of

- a trivial representation and a sign representation,
- a 3-dimensional representation, constructed as above,
- the *tensor* of two representations,
- orthogonality and sum of squares formula.

χ	e $\# = 1$	(12) $\# = 3$	$(12)(34)$ $\# = 3$	(123) $\# = 8$	(1234) $\# = 6$
trivial	1	1	1	1	1
sign	1	-1	1	1	-1
$\mathbb{C}^3$	3	1	-1	0	-1
$\mathbb{C}^3 \otimes$ sign	3	-1	-1	0	-1
$(\mathbb{C}^2)$	2	0	2	-1	0

Now what is this last 2-dimensional representation? Recall the last row of the character table of $\mathfrak{S}_3$: they coincide. Actually, this representation of $\mathfrak{S}_4$ is a *pullback* of that representation of $\mathfrak{S}_3$.

Consider finite groups G, H and a surjective homomorphism $\varphi : G \to H$. Then for any irreducible representation V of H, the composition $G \to H \to \mathrm{GL}(V)$ is a representation of G, and is irreducible by the density theorem. This representation is called the **pullback representation**.

Now consider $\mathfrak{A}_4$. There is a surjective homomorphism $\mathfrak{A}_4 \to \mathbb{Z}_3$ by quotienting out $(12)(34)$. Therefore we get the trivial representation and two pullback representations:

χ	e $\# = 1$	$(12)(34)$ $\# = 3$	(123) $\# = 4$	(132) $\# = 4$
trivial	1	1	1	1
pullback	1	1	ζ	ζ^2
pullback	1	1	ζ^2	ζ
$(\mathbb{C}^3)$	3	0	0	-1

The last row is by **restriction**: if H is a subgroup of G, then $H \to G \to \mathrm{GL}(V)$ is a representation of H.

3.2.2 Frobenius Determinant

For a finite group G and formal variables x_g for $g \in G$, define a $|G| \times |G|$ matrix X_G by defining the entry at row g column h to be $x_{g^{-1} \cdot h}$.

THM 3.7 Suppose G is a finite group, not necessarily abelian. Then

$$\det X_G = \prod_{j=1}^{r} p_j(\underline{x})^{\deg p_j}, \tag{19}$$

where p_j are pairwise non-proportional irreducible polynomials and r is the number of conjugacy classes of G.

3.2.3 Frobenius-Schur Indicator

Let G be a finite group and V a finite dimensional complex representation.

Say V is of

- **complex type** or **$\mathbb{C}$-type** if V is not representation isomorphic to its dual representation V^*.
- **real type** or **$\mathbb{R}$-type** if V admits a non-degenerate symmetric G-invariant bilinear form.
- **quaternionic type** or **$\mathbb{H}$-type** if V admits a non-degenerate anti-symmetric G-invarient bilinear form.

Every finite dimensional complex representation V is either of complex type, or of real type (in which case $V = U \otimes_{\mathbb{R}} \mathbb{C}$ for some *real* representation R), or of quaternionic type.

THM 3.8

For a representation V, let the **Frobenius-Schur inducator** be

$$\mathrm{FS}(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^2). \tag{20}$$

Then

$$\mathrm{FS}(V) = \begin{cases} 1 & \text{if } V \text{ is of } \mathbb{R}\text{-type}, \\ -1 & \text{if } V \text{ is of } \mathbb{H}\text{-type}, \\ 1 & \text{if } V \text{ is of } \mathbb{C}\text{-type}. \end{cases} \tag{21}$$

Proof. For a matrix A, $\operatorname{tr} A^2 = \operatorname{tr}_{S^2 V}(A \otimes A) - \operatorname{tr}_{\Lambda^2 V}(A \otimes A)$. Therefore, letting p be the symmetrizer,

$$\mathrm{FS}(V) = \chi_{S^2 V}(p) - \chi_{\Lambda^2 V}(p) = \dim \left(S^2 V\right)^G - \dim \left(\Lambda^2 V\right)^G. \tag{22}$$

Now notice that $\mathrm{Hom}(V^*, V) \cong (V \otimes V)^G \cong \left(S^2 V\right)^G \oplus \left(\Lambda^2 V\right)^G$. The dimensions of the direct summand could be $0, 0$, $1, 0$ or $0, 1$, depending on whether V is of $\mathbb{C}$-type, $\mathbb{R}$-type or $\mathbb{H}$-type. □

3.2.4 Frobenius Divisibility

Denote by $\mathbb{A}$ the ring of algebraic integers.

THM 3.9 Frobenius divisibility theorem

If G is a finite group, V is a complex irreducible representation, then $\dim V \mid |G|$.

Proof.

LEM 3.10

Suppose C is a conjugacy class in G. Then $\lambda := \chi_V(C) \cdot \frac{|C|}{\dim V}$ is an algebraic integer.

Proof of Lemma. Let $p_C = \sum_{g \in C} g$. Schur's lemma implies that p_C acts on V as a scalar μ, so $\operatorname{tr} p_C = \mu \dim V$. Therefore, $\mu = \chi_V(C) \cdot \frac{|C|}{\dim V} = \lambda$.

$\mathbb{Z}[G]$ is a finitely generated $\mathbb{Z}$-module, and $\mathbb{Z}$ is Noetherian; so any submodule of $\mathbb{Z}[G]$ is Noetherian. In particular, for any $x \in \mathbb{Z}[G]$, $\mathbb{Z}[x]$ is finitely generated. Therefore, there is some n such that $\mathbb{Z}[x]$ is generated by $1, x, \cdots, x^n$. Therefore $x^{n+1} = a_n x^n + \cdots + a_0$. Taking $x = p_C$ and evaluate on V, we obtain $\lambda^{n+1} = a_n \lambda^n + \cdots + a_0$, so λ is an algebraic integer. □

$\mathbb{Q} \ni \frac{|G|}{\dim V} = \frac{|G|}{\dim V} \langle \chi_V, \chi_V \rangle = \frac{1}{\dim V} \sum_C |\chi_V(C)|^2 |C| = \sum_C \lambda_C \overline{\chi_V(C)}$, where λ_C defined as in the Lemma. The right hand side is an algebraic integer, so $\frac{|G|}{\dim V} \in \mathbb{Q} \cap \mathbb{A} = \mathbb{Z}$. □

COR 3.11 Actually, $\dim V$ divides $\frac{|G|}{|Z(G)|}$.

3.2.5 Burnside's Theorem

THM 3.12 Let G be a finite group. If a conjugacy class C of G has p^a elements (where p is a prime and $a > 0$), then G is not simple.

Proof. The irreducible representations of G can be classified into three types: the trivial representation; non-trivial irreducible representations with dimension divisible by p, denoted by D; non-trivial irreducible representations with dimension not divisible by p, denoted by N. Then $\sum_{V \text{ irr.}} \chi_V(C) \dim V = 0$, so $b = \sum_{V \in D} \frac{1}{p} \cdot \chi_V(C) \dim V \in \mathbb{A}$. So

$$0 = 1 + pb + \sum_{V \in N} \chi_V(C) \dim V. \tag{23}$$

However, $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$ and $b \in \mathbb{A}$, so $\chi_V(C)$ is a non-zero algebraic integer for some $V \subseteq N$. $\chi_V(C)$ is the sum of some roots of unity; if the average of some roots of unity is an algebraic integer, then either they sum to 0, or they are all equal. Therefore, $\rho_V(C) = \lambda \cdot \mathrm{id}$, and $\ker \rho_V$ is a non-trivial normal subgroup of G. □

COR 3.13
Burnside
If $|G| = p^a q^b$ for prime p and q, then G is solvable.

3.3 Virtual Representations

A **virtual representation** of a group G is a formal linear combination of representations $\sum_i n_i V_i$ where $n_i \in \mathbb{Z}$. The character of a virtual representation is the linear combination of the corresponding characters. If the inner product of a virtual character is 1 and the total dimension is non-zero, then this virtual character is a single irreducible representation.

3.4 Restriction and Induction

If G is a finite group, V a representation of G and H is a subgroup of G, then the representation $\rho_V : G \to \mathrm{GL}(V)$ restricts to a representation of H: $H \hookrightarrow G \to \mathrm{GL}(V)$. This representation is called the **restriction**, denoted $\mathrm{Res}_H^G(V)$. Equivalently, $\mathrm{Res}_H^G(V) \cong \mathrm{Hom}_{k[G]}(k[G], V)$ where $k[G]$ is seen as a $(k[G], k[H])$ -bimodule; or $k[G] \otimes_{k[G]} V$, where $k[G]$ is seen as a $(k[H], k[G])$-bimodule.

The induced representation is the adjoint of the restricted representation. By the tensor-Hom adjoint pair, we have, for a left B-module N and a left A-module M,

$$\begin{aligned} \mathrm{Hom}_A(A \otimes_B N) &\cong \mathrm{Hom}_B(N, M), \\ \mathrm{Hom}_A(N, M) &\cong \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, M)). \end{aligned} \tag{24}$$

The module $A \otimes_B N$ is called the **induced A-module** from N, and $\mathrm{Hom}_A(B, M)$ is called the **coinduced B-module** from M.

Now suppose U is a representation of H. Then U is a left $k[H]$-module; we let the **induced representation** of U be the induced $k[G]$-module $\mathrm{Ind}_H^G(U) = k[G] \otimes_{k[H]} U$, and the **coinduced representation** of be the coinduced $k[G]$ -module $\mathrm{coInd}_H^G(U) = \mathrm{Hom}_{k[H]}(k[G], U)$. For finite groups, the induced representation is isomorphic to the coinduced representation.

PROP 3.14

1. If $K \subseteq H \subseteq K$, then $\mathrm{Ind}_K^H \mathrm{Ind}_H^G = \mathrm{Ind}_K^G$.
2. If U is a representation of $H \subseteq G$, then

$$\mathrm{Ind}_H^G(U) \cong \{f : G \to U : f(hg) = \rho_U(h) f(g), \forall g \in G, h \in H\}. \tag{25}$$

   The action is given by $(g \cdot f)(u) = f(xg)$ for x and $g \in G$.
3. $\dim \mathrm{Ind}_H^G(U) = \dim U \cdot \frac{|G|}{|H|}$.

As a vector space, $\mathrm{Ind}_H^G(V)$ can be written as $\bigoplus_i g_i \otimes U$, where g_i are coset representatives of H.

THM 3.15 Frobenius reciprocity

Suppose H is a subgroup of a finite group G, W and representation of H and V a representation of G. Then

$$\begin{aligned} \mathrm{Hom}_G\left(\mathrm{Ind}_H^G(U), V\right) &\cong \mathrm{Hom}_H\left(U, \mathrm{Res}_H^G(V)\right); \\ \mathrm{Hom}_G\left(V, \mathrm{Ind}_H^G(U)\right) &\cong \mathrm{Hom}_H\left(\mathrm{Res}_H^G(V), U\right). \end{aligned} \tag{26}$$

In other words, Res and Ind are adjoint functors.

For complex representations, if χ and ψ are two characters (or in general class functions) of H and G respectively,

$$\left(\chi, \operatorname{Res}_H^G \psi\right)_H = \left(\operatorname{Ind}_H^G \chi, \psi\right)_G. \tag{27}$$

We also have the formula

$$\operatorname{Ind}_H^G\left(\varphi \cdot \operatorname{Res}_H^G(\psi)\right) = \operatorname{Ind}_H^G(\varphi) \cdot \psi. \tag{28}$$

THM 3.16 Frobenius character formula

Suppose H is a subgroup of G, char $k \nmid |H|$, U is a representation of H and $V = \operatorname{Ind}_H^G(U)$. Then

$$\chi_V(x) = \sum_{g_i x g_i^{-1} \in H} \chi_U\left(g_i x g_i^{-1}\right) = \frac{1}{|H|} \sum_{\substack{g \in G \\ gxg^{-1} \in H}} \chi_U(gxg^{-1}), \tag{29}$$

where g_i are coset representatives of H.

3.4.1 Mackey's Irreducibility Criterion

Introduce the **double cosets**: if H and K are two subgroups of G, then G is the disjoint union of some KsH for some double coset representatives s. The set of double cosets (or a system of their representatives) may be denoted as $K \setminus G / H$.

Suppose H and K are subgroups of G and (W, ρ) is a representation of H. We wish to study $\operatorname{Res}_K^G\left(\operatorname{Ind}_G^H(W)\right)$.

$V = \operatorname{Ind}_G^H(W)$ is the direct sum of images xW for $x \in G / H$. Let $s \in K \setminus G / H$ and let $V(s)$ be the subspace of V generated by the images yW for $y \in KsH$. Then $\operatorname{Res}_K^G V \cong V$ as a vector space is also a direct sum of the $V(s)$'s and $V(s)$ is stable under K, so they are also isomorphic as representations of K.

We can further decompose $V(s)$: the subgroup of K consisting of elements x such that $x(sW) = sW$ is equal to $H_s = K \cap sHs^{-1}$, so $V(s)$ is a direct sum of the images $x(sW)$ for $x \in K / H_s$. This is to say $V(s) = \operatorname{Ind}_{H_s}^K(sW)$. The representation sW is in turn isomorphic to (W_s, ρ^s) where $\rho^s(x) = \rho(s^{-1}xs)$. Therefore,

THM 3.17 Mackey's formula

$\operatorname{Res}_K^G\left(\operatorname{Ind}_H^G(W)\right) \cong \bigoplus_{s \in S} \operatorname{Ind}_{H_s}^K(W_s)$ as representations of K.

Taking $K = H$, we derive an irreducibility criterion for induced representations. Two representations W_1, W_2 of G are called **disjoint** if $\operatorname{Hom}_G(W_1, W_2) = 0$.

COR 3.18 Mackey's irreducibility criterion

The induced representation $V = \operatorname{Ind}_H^G W$ is irreducible, if and only if W is irreducible, and the two representations ρ^s and $\operatorname{Res}_{H_s}^H(\rho)$ are disjoint. Here $H_s = H \cap sHs^{-1}$ and $\rho^s(x) = \rho\left(s^{-1}xs\right)$.

COR 3.19

If $H \lhd G$, then $\operatorname{Ind}_H^G(\rho)$ is irreducible if and only if ρ is irreducible and not isomorphic to any of its conjugates ρ^s for $s \notin H$.

EX 3.20 normal subgroups

Let A be a normal subgroup of a group G and let (V, ρ) be an irreducible representation of G. Let $V = \bigoplus V_i$ be the isotypical decomposition of $\operatorname{Res}_A^G(V)$. For $s \in G$, $\rho(s)$ permutes the V_i; since V is irreducible it permutes transitively.

Let V_{i_0} be one of these V_i's; if $V_{i_0} = V$ then $\mathrm{Res}^G_A(V)$ is isotypic, i.e. a direct sum of isomorphic irreducible representations. Otherwise let H be the subgroup of G consisting of $s \in G$ such that $\rho(s)V_{i_0} = V_{i_0}$. Then $A \leqslant H < G$ and $\mathrm{Res}^G_A(V)$ is induced by the natural representation of H on V_{i_0}. Therefore,

PROP 3.21 If $A \lhd G$, then

- either there is a subgroup $A \leqslant H < G$ and an irreducible representation σ of H such that ρ is induced by σ; or
- $\mathrm{Res}^G_A(\rho)$ is isotypic.

EX 3.22 semidirect products by an abelian group

Suppose $G = A \rtimes H$ where A is abelian. We wish to construct irreducible representations of G from certain subgroups of H. (This is the method of *little groups* of Wigner and Mackey).

Since A is abelian, its irreducible representations are of dimension 1 and they form a group $\hat{A} = \mathrm{Hom}(A, \mathbb{C}^\times)$. G acts on $\hat{A}$ as $(s\chi)(a) = \chi(s^{-1}as)$.

Let χ_i $(i \in \hat{A} / H)$ be a system of representatives for the orbits of H in $\hat{A}$. For each $i \in \hat{A} / H$ let H_i be the stabilizer subgroup of χ_i, and let $G_i = A \rtimes H_i$ be the corresponding subgroup of G. Extend the functions χ_i to G_i by $\chi_i(ah) = \chi_i(a)$ for $a \in A$ and $h \in H_i$. Since h fixes χ_i, χ_i is a 1-dimensional character of G.

Now let ρ be an irreducible representation of H_i. By composing ρ with the projection $G_i \to H_i$ we obtain as irreducible representation $\tilde{\rho}$ of G_i. Take the tensor product of χ_i and $\tilde{\rho}$; we thus obtain an irreducible representation $\chi_i \otimes \tilde{\rho}$ of G_i. Let

$$\theta_{i,\rho} = \mathrm{Ind}^G_{G_i}(\chi_i \otimes \tilde{\rho}). \tag{30}$$

PROP 3.23

1. $\theta_{i,\rho}$ is irreducible.
2. $\theta_{i,\rho}$ and $\theta_{i',\rho'}$ are isomorphic, if and only if $i = i'$ and $\rho = \rho'$.
3. Every irreducible representation of G is isomorphic to some $\theta_{i,\rho}$.

3.4.2 Artin's Theorem

Recall the set of class functions $F_c(G, \mathbb{C})$ and the inner product on it. Induction gives a map

$$\begin{aligned} \mathrm{Ind}^G_H &: F_c(H, \mathbb{C}) \to F_c(G, \mathbb{C}), \\ f &\mapsto \left(x \mapsto \frac{1}{|H|} \sum_{\substack{g \in G, \\ gxg^{-1} \in H}} f(gxg^{-1}) \right). \end{aligned} \tag{31}$$

Let $\chi_1, \cdots, \chi_n$ be the set of irreducible characters of G and let $R(G)$ be the ring of virtual characters,

$$R(G) = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n. \tag{32}$$

Ind_H^G and Res_H^G induce ring homomorphisms $\mathrm{Ind} : R(H) \to R(G)$ and $\mathrm{Res} : R(G) \to R(H)$, respectively. They are adjoints with respect to the bilinear forms $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_G$,; moreover because $\mathrm{Ind}(\varphi \cdot \mathrm{Res}(\psi)) = \mathrm{Ind}(\varphi) \cdot \psi$, the image of Ind is an ideal of $R(G)$.

THM 3.24
Artin's theorem

Let $\mathcal{X}$ be a family of subgroups of G and $\mathrm{Ind} : \bigoplus_{H\in\mathcal{X}} R(H) \to R(G)$. Then G is the union of all conjugates of the subgroups of $\mathcal{X}$, if and only if the cokernel of Ind is finite.

Since $R(G)$ is finitely generated as an abelian group, the latter condition may be restated as, for every character χ of G, there are virtual characters $\chi_H \in R(H)$ for $H \in \mathcal{X}$ and $d \in \mathbb{Z}_+$ such that $d \cdot \chi = \sum_{H\in\mathcal{X}} \mathrm{Ind}_H^G(\chi_H)$.

Notice that the family of cyclic subgroups of G satisfies the first condition; therefore, as a corollary,

COR 3.25

Every character of G is a linear combination with rational coefficients of characters induced by characters of cyclic subgroups of G.

First proof of Artin's theorem. The first condition is equivalently stated as $\mathrm{Ind} \otimes_{\mathbb{Z}} \mathbb{C} : \bigoplus_{H\in\mathcal{X}} R(H) \otimes_{\mathbb{Z}} \mathbb{C} \to R(G) \otimes_{\mathbb{Z}} \mathbb{C}$ is surjective, or by adjointness, $\mathrm{Res} \otimes_{\mathbb{Z}} \mathbb{C}$ is injective. If the first condition holds, then any class function on G which restricts to zero on each subgroup in $\mathcal{X}$ is itself zero because G is covered by conjugates of subgroups in $\mathcal{X}$, so injectivity holds.

Conversely if the second condition holds let S be the union of conjugates of the subgroups in $\mathcal{X}$. Every class function on G is of the form $\sum_{H\in\mathcal{X}} \mathrm{Ind}_H^G(f_H)$, so it vanishes outside of S. Therefore the complement of S must be empty, so $S = G$ is the union of all conjugates of subgroups in $\mathcal{X}$. □

Second proof of Artin's theorem, "only if" part. If C is a cyclic group, let θ_A be a class function on A defined by

$$\theta_A(x) = \begin{cases} a \text{ if } x \text{ genereates } A \\ 0 \text{ otherwise} \end{cases} \tag{33}$$

Claim. If G is a finite group, then $|G| = \sum_{C\leqslant G \text{ cyclic}} \mathrm{Ind}_C^G(\theta_C)$.

The claim can be proved by Theorem 3.16. Also, this proves $\theta_C = |C| - \sum_{B<C \text{ cyclic}} \mathrm{Ind}_B^A(\theta_B)$, so by induction on $|C|$, $\theta_C \in R(C)$. Therefore, the constant function $|G|$ is in the image of Ind. Since the image of Ind is an ideal, $\mathrm{im}\,\mathrm{Ind}$ contains every element of the form $g\chi$, so coker Ind is finite (actually its order is a factor of $|G|$). □

4

Section 4 Representations of Symmetric Groups

4.1 Young Tableux and Symmetric Functions

For materials on Young tableux and symmetric functions we refer to the book *Young Tableux with Applications to Representation Theory and Geometry*, especially section 6 *Symmetric Functions*, and also the corresponding notes. We fix some notation here.

The graded ring of symmetric functions on n variables is denoted by $\Lambda_n = \bigoplus_{r\geqslant 0}\Lambda_n^r$, and $\Lambda = \varprojlim_n \Lambda_n$.

The **monomial symmetric function** is $m_\lambda(x_1,\cdots,x_n) = \sum_\alpha x^\alpha \in \Lambda_n^r$ where $\lambda \vdash r$ and α varies over all permutations of λ. $m_{(1^r)}$ is called the **elementary symmetric function** e_r, and $e_\lambda = e_{\lambda_1}\cdot e_{\lambda_2}\cdot\cdots$. The **complete symmetric function** is $h_r(x_1,\cdots,x_n) = \sum_{\lambda\vdash r} m_\lambda$ and $h_\lambda = h_{\lambda_1}\cdot h_{\lambda_2}\cdot\cdots$. The **power sum** is $p_r = m_{(r)}$, and $p_\lambda = p_{\lambda_1}\cdot p_{\lambda_2}\cdot\cdots$. Let s_λ be the **Schur polynomial**. These are all $\mathbb{Z}$-bases for Λ.

An inner product on Λ is defined by $\left\langle s_\lambda, s_\mu\right\rangle = \begin{cases}1 \text{ if } \lambda=\mu\\ 0 \text{ otherwise}\end{cases}$. In this inner product, h_λ is dual to m_λ, and p_λ is dual to $\frac{p_\lambda}{z(\lambda)}$, where $z(\lambda) = \prod_r r^{m_r}\cdot m_{r!}$, and m_r is the number of times r appears in λ. We obtain the following three identities:

$$\prod_{i=1}^{m}\prod_{j=1}^{l}\frac{1}{1-x_iy_j} = \sum_\lambda h_\lambda(x)m_{\lambda(y)} = \sum_\lambda \frac{1}{z(\lambda)}p_\lambda(x)p_\lambda(y) = \sum_\lambda s_\lambda(x)s_\lambda(y). \tag{34}$$

$\omega : s_\lambda \mapsto s_{\lambda^\top}$ is a ring automorphism, and also an involution and an isometry (i.e. preserving the inner product). Under this involution, $\omega(h_\lambda) = e_\lambda$, $\omega(p_\lambda) = (-1)^{\sum(p_i-1)} p_\lambda$.

The *Jacobi-Trudy identity* is

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1\leqslant i,j\leqslant n}, \tag{35}$$

where $h_r = 0$ for $r < 0$. The dual identity (by applying ω) is

$$s_\lambda = \det\left(e_{\lambda_i^\top - i + j}\right)_{1\leqslant i,j\leqslant n}. \tag{36}$$

4.2 Specht Modules

Let T denote a Young diagram of shape $\lambda \vdash n$ numbered by integers 1 to n with no repeats (not necessarily a Young tableau). The action of $\mathfrak{S}_n$ on T, $\sigma \cdot T$, puts number $\sigma(i)$ in the box in which T puts i. The **column / row group** of T is the subgroup of $\mathfrak{S}_n$ that permutes each column / row within themselves, denoted C_T and R_T.

Define

$$b_T = \sum_{g\in C_T} (-1)^g g,\; a_T = \sum_{g\in R_T} g,\; c_T = b_T a_T \in \mathbb{C}[\mathfrak{S}_n]. \tag{37}$$

Then for any $r \in R_T$ and $c \in C_T$, $b_T r = r b_T = (-1)^r b_T$ and $c a_T = a_T c = a_T$. In particular, $a_T^2 = |R_T| a_T$ and $b_T^2 = |C_T| b_T$.

THM 4.1 $V_\lambda := \mathbb{C}[\mathfrak{S}_n] \cdot c_U$ (where U is any numbered Young diagram of shape λ) is an irreducible representation of $\mathfrak{S}_n$, and every irreducible representation of $\mathfrak{S}_n$ is isomorphic to some V_λ. These representations are called **Specht modules**.

As a corollary all (complex) representations of $\mathfrak{S}_n$ can be realized over $\mathbb{Q}$.

Proof. Define a ***tabloid*** $\{T\}$ as the orbit of a numbered Young diagram T under the action of the row group, and let M_λ be the space with basis all tabloids. We can make the identification that $\sigma \in \mathfrak{S}_n$ corresponds to $\sigma \cdot U$; then M_λ is a left $\mathbb{C}[\mathfrak{S}_n]$-module, and V_λ is spanned by all $\sigma \cdot b_U \cdot \{U\}$ where $\sigma \in \mathfrak{S}_n$. Denote $v_T = b_T \cdot \{T\}$, then $\sigma \cdot v_T = v_{\sigma\cdot T}$; so V_λ is stable under $\mathbb{C}[\mathfrak{S}_n]$, and in fact V_λ is essentially the subspace of M_λ spanned by $b_T \cdot \{T\}$ for all numbered Young diagrams T of shape λ.

LEM 4.2

1. Suppose T_λ and $T'_{\lambda'}$ are two numbered Young diagrams and λ does not strictly dominate λ'. Then either there are two distinct integers that occur in the same row of T' and the same column of T, or $\lambda = \lambda'$ and there is some $p' \in Q_{\lambda'}$ and $q \in P_\lambda$ such that $p' \cdot T' = q \cdot T$. (The two cannot happen at the same time.)
2. If there is a pair of integers in the same row of T' and in the same column of T, then $b_T \cdot \{T'\} = 0$. Otherwise, $b_T \cdot \{T'\} = \pm v_T$.

As a corollary,

$$\begin{aligned} b_T \cdot M_\lambda = b_T \cdot V_\lambda = \mathbb{C} v_T \neq 0, \\ b_T \cdot M_{\lambda'} = b_T \cdot V_{\lambda'} = 0 \text{ if } \lambda' > \lambda. \end{aligned} \tag{38}$$

The $\lambda' > \lambda$ (lexicographic order) condition can be relaxed to $\lambda \not\trianglerighteq \lambda'$ (dominance order).

This proves irreducibility: any subrepresentation of V_λ must contain v_T under the action of b_T, and any subrepresentation containing v_T generates V_λ under the action of $\mathfrak{S}_n$. Moreover they are pairwise non-isomorphic by the second equation. Since the number of irreducible representations of $\mathfrak{S}_n$ is equal to the number of conjugacy classes, which is in turn equal to the number of partitions of n, we have thus produced all irreducible representations of $\mathfrak{S}_n$. □

Since the exact U defining V_λ does not matter, we choose any numbered diagram U and denote c_U by c_λ.

Equation 38 has an important consequence that

PROP 4.3 For any $x \in \mathbb{C}[\mathfrak{S}_n]$, there is some scalar $\mu \in \mathbb{C}$ such that $b_\lambda x a_\lambda = \mu b_\lambda a_\lambda$. If $\lambda' > \lambda$ (or, actually, $\lambda \not\trianglerighteq \lambda'$), then $b_\lambda x a_{\lambda'} = 0$.

The proof is just by Equation 38 and apply the identification $\sigma \longleftrightarrow \sigma \cdot U$.

PROP 4.4 $c_\lambda^2 = \frac{n!}{\dim V_\lambda} c_\lambda$.

Proof. By the previous proposition $c_\lambda^2 = n_\lambda c_\lambda$. Let φ be the right multiplication by $\frac{c_\lambda}{n_\lambda}$. Then $\operatorname{tr} \varphi = \frac{n!}{n_\lambda}$. However, $\frac{c_\lambda}{n_\lambda}$ is an idempotent, so φ is a projection onto $\operatorname{im} \varphi$. So $\operatorname{tr} \varphi = \dim \mathbb{C}[\mathfrak{S}_n] \frac{c_\lambda}{n_\lambda} = \dim V_\lambda$. Therefore, $n_\lambda = \frac{n!}{\dim V_\lambda}$. □

Recall that M_λ is the space with basis all tabloids of shape λ. Again by Equation 38,

PROP 4.5 $\operatorname{Hom}_{\mathbb{C}[\mathfrak{S}_n]}\left(M_\lambda, V_\mu\right) = 0$ if $\mu \not\trianglerighteq \lambda$. $\operatorname{Hom}_{\mathbb{C}[\mathfrak{S}_n]}(M_\lambda, V_\lambda) = 1$.

In particular $M_\lambda = V_\lambda \oplus \bigoplus_{\mu \triangleright \lambda} k_{\mu\lambda} V_\mu$.

Proof. We need an algebraic lemma:

LEM 4.6 If A is an algebra with unit and an idempotent e, M a left A-module, then $\operatorname{Hom}_A(Ae, M) \cong eM$.

Using this lemma, let $A = \mathbb{C}[\mathfrak{S}_n]$, $e = a_\lambda$ (not quite an idempotent but up to a scalar) and $M = V_\mu$, we have $\mathrm{Hom}_{\mathfrak{S}_n}(M_\lambda, V_\mu) = \mathrm{Hom}_A(Aa_\lambda, Ac_\mu) \cong a_\lambda A b_\mu a_\mu$. If $\mu \not\trianglerighteq \lambda$, then $a_\lambda A b_\mu = 0$ (Proposition 4.3, inverse); therefore $\mathrm{Hom}_{\mathfrak{S}_n}(M_\lambda, V_\mu) = 0$. □

We will prove that the numbers $k_{\mu\lambda}$ are actually the Kostka numbers.

Denote χ_{M_λ} the character of the representation of $\mathfrak{S}_n$ on M_λ, and C_μ the conjugacy class of $\mathfrak{S}_n$ defined by the partition μ. Then $\chi_{M_\lambda}(C_\mu)$ is the number of tabloids fixed by σ. By a binomial expansion, we can show that

PROP 4.7 The character χ_{M_λ} on the conjugacy class C_μ is equal to the coefficient before m_λ in p_μ.

That is,

$$p_\mu(x) = \sum_{\lambda \vdash n} \chi_{M_\lambda}(C_\mu) \cdot m_\lambda(x), \tag{39}$$

or by dualizing,

$$h_\lambda(x) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_{M_\lambda}(C_\mu) p_\mu(x). \tag{40}$$

REM 4.8 The convention in the class and Etingof's book is different: they used $c_\lambda = a_\lambda b_\lambda$ instead of $b_\lambda a_\lambda$ here. The resulting representations are isomorphic.

4.3 Ring of Representations

Let R_n be the Grothendieck group of $\mathfrak{S}_n$, the free abelian group with basis given by isomorphism classes of finite dimensional representations of $\mathfrak{S}_n$, such that $[V] + [W] = [U]$ if $U \cong V \oplus W$. Let $R = \bigoplus_{n \geqslant 0} R_n$.

V_λ forms a basis of R. Since the relation between M_λ and V_ν is upper-triangular (Proposition 4.5), $[M_\lambda]$ is also a basis of R.

We can define a multiplication on R as follows. For $[V] \in R_m$ and $[W] \in R_n$, consider $\mathfrak{S}_m \times \mathfrak{S}_n$ as a subgroup of $\mathfrak{S}_{m+n}$, then $[V] \circ [W] := \left[\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} V \otimes W\right]$. This R is called the **representation ring** of $\mathfrak{S}_n$.

This ring R is a commutative, associative, graded ring with unit, with a bilinear form $([V],[W]) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi_V(\sigma) \overline{\chi_W(\sigma)}$, and an involution $[V] \mapsto [V \otimes \mathrm{sign}]$.

We show that there is a nice correspondence φ between R and the ring of symmetric functions Λ; "*nice*" in terms that φ is an isomorphism, an isometry (preserving the bilinear forms) and φ preserves the involution.

This φ is defined by $\varphi : \Lambda \to R$, $h_\lambda \mapsto [M_\lambda]$.

PROP 4.9

1. This φ is a ring isomorphism and an isometry.
2. $\varphi(s_\lambda) = [V_\lambda]$.
3. If $\omega : \Lambda \to \Lambda$ is the involution $h_\lambda \mapsto e_\lambda$ and $\omega : R \to R$ is the involution $[V] \mapsto [V \otimes \mathrm{sign}]$, then $\varphi \circ \omega = \omega \circ \varphi$.

Proof. $\Lambda = \mathbb{Z}[h_1, h_2, \cdots]$. Therefore, to verify that φ is a ring homomorphism we only need to prove, for $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k)$, that $\left[M_{\lambda_1}\right] \circ \cdots \circ \left[M_{\lambda_k}\right] = [M_\lambda]$. This follows from the fact that $[M_\lambda] = \mathrm{Ind}_{R_\lambda}^{\mathfrak{S}_n} 1$. Since $[M_\lambda]$ forms a basis of R, φ is a ring isomorphism.

Recall that $h_\lambda = \sum_\mu \frac{1}{z_\mu} \chi_{M_\lambda}(C(\mu)) p_\mu$. Construct an inverse $\psi : R \to \Lambda \otimes \mathbb{Q}$ by $[V] \mapsto \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(C(\mu)) p_\mu$. Then the composition $\Lambda \xrightarrow{\varphi} R \xrightarrow{\psi} \Lambda \otimes \mathbb{Q}$ is $h_\lambda \mapsto [M_\lambda] \mapsto \sum_\mu \frac{1}{z_\mu} \chi_{M_\lambda}(C(\mu)) p_\mu = h_\lambda$ is the identity map. Therefore ψ is the inverse of φ and in particular the image of ψ lies in Λ itself.

We next show φ is an isometry. To do this it's enough to show that ψ is an isometry, or $([V], [W]) = \langle \psi([V]), \psi([W]) \rangle$. Recall that $\left\langle p_\lambda, p_\mu \right\rangle = \begin{cases} z_\lambda = z_\mu & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$. So

$$
\begin{aligned}
\langle \psi([V]), \psi([W]) \rangle &= \sum_{\lambda \vdash n} \sum_{\mu \vdash n} \frac{1}{z_\lambda z_\mu} \chi_V(C(\lambda)) \chi_W(C(\lambda)) \left\langle p_\lambda, p_\mu \right\rangle \\
&\overset{\lambda=\mu}{=\!=} \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(C(\mu)) \chi_W(C(\mu)) \\
&\overset{z_\lambda = \frac{n!}{|C(\lambda)|}}{=\!=\!=} \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} \chi_V(g) \chi_W(g) \\
&\overset{V \text{ and } W \text{ are real}}{=\!=\!=\!=} ([V], [W]).
\end{aligned}
\tag{41}
$$

Recall that $h_\lambda = s_\lambda + \sum_{\nu > \lambda} K_{\nu\lambda} S_\nu$, where $K_{\nu\lambda}$ are the Kostka numbers. Since this is an upper-triangular relation, we can solve s_λ in terms of h_λ: $s_\lambda = h_\lambda + \sum_{\nu > \lambda} a_{\nu\lambda} h_\nu$. Apply φ: $\varphi(s_\lambda) = [M_\lambda] + \sum_{\nu > \lambda} a_{\nu\lambda} [M_\nu] = [V_\lambda] + \sum_{\nu > \lambda} b_{\nu\lambda} [V_\nu]$. Since φ is an isometry, it follows that $1 = \langle s_\lambda, s_\lambda \rangle = (\varphi(s_\lambda), \varphi(s_\lambda)) = 1 + \sum b_{\nu\lambda}^2$. Therefore, $b_\nu \lambda = 0$ and the right hand side must be $[V_\lambda]$ alone.

To prove the third part, we prove that $\psi \circ \omega = \omega \circ \psi$. For any representation V of $\mathfrak{S}_n$,

$$
\begin{aligned}
\psi(V \otimes \mathrm{sign}) &= \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(C(\mu)) (-1)^{\sum_k (\mu_k - 1)} p_\mu \\
&= \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(C(\mu)) (-1)^{|\mu| - l(\mu)} p_\mu \\
&= \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(C(\mu)) \omega\left(p_\mu\right) \\
&= \omega(\psi(V)).
\end{aligned}
\tag{42}
$$

□

In particular, $M_\lambda = V_\lambda \oplus \bigoplus_{\mu \rhd \lambda} K_{\mu\lambda} V_\mu$.

With this result, we can transfer our knowledge about symmetric functions to representations of $\mathfrak{S}_n$, and *vice versa*. For example, $p_\mu(x) = \sum_{\lambda \vdash n} \chi_{V_\lambda}(C(\mu)) s_\lambda(x)$. For another example, recall the Littlewood-Richardson coefficients $s_\lambda s_\mu = \sum_{\nu \vdash |\lambda|+|\nu|} c^\nu_{\lambda\mu} s_\nu$. By interpreting $c^\nu_{\lambda\mu}$ as a multiplicity of irreducible representation, we can proof they are non-negative integers.

If V is a representation of $\mathfrak{S}_n$, define its **Frobenius character** by $F_V(x) := \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(C(\mu)) p_\mu(x) \in \Lambda$. Then

COR 4.10 $F_{M_\lambda}(x) = h_\lambda(x)$, $F_{V_\lambda}(x) = s_\lambda(x)$.

PROP 4.11 Frobenius character formula

If $\lambda \vdash n$, $N \geqslant l(\lambda)$ ($l(\lambda)$ denoting the length of λ), then

$$\chi_{V_\lambda}(C(\mu)) = \left[\prod_{i=1}^{N} x_i^{\lambda_i+N-i}\right]\left(\prod_{1\leqslant i<j\leqslant N}(x_i - x_j) p_\mu(x_1, \cdots, x_N)\right). \tag{43}$$

$[x^\lambda](p(x))$ denotes the coefficient in $p(x)$ before x^λ.

Proof. We first assume that $N \geqslant n$ and let $x_{N+1} = x_{N+2} = \cdots = 0$. Then

$$p_\mu(x_1, \cdots, x_N) = \sum_{\lambda \vdash n} \chi_{V_\lambda}(C(\mu)) s_\lambda(x_1, \cdots, x_N). \tag{44}$$

For $\alpha \in \mathbb{Z}^N_{\geqslant 0}$, define $a_\alpha(x) = \det\left(x_i^{\alpha_j}\right)_{1\leqslant i,j\leqslant N}$. Let $\delta = (N-1, N-2, \cdots, 0)$. Then $s_\lambda(x_1, \cdots, x_N) = \frac{a_{\lambda+\delta}(x)}{a_\delta(x)}$.

Since $a_\delta(x) = \prod_{1\leqslant i<j\leqslant N}\left(x_i - x_j\right)$,

$$\prod_{1\leqslant i<j\leqslant N}\left(x_i - x_j\right) \cdot p_\mu(x_1, \cdots, x_N) = \sum_{\lambda \vdash n} \chi_{V_\lambda}(C(\mu)) a_{\lambda+\delta}(x). \tag{45}$$

To proof the proposition, we only need to observe that for $\lambda, \nu \vdash n$,

$$[x^{\lambda+\delta}](a_{\nu+\delta}(x)) = \begin{cases} 1 \text{ if } \nu = \lambda \\ 0 \text{ if } \nu \neq \lambda' \end{cases} \tag{46}$$

which directly follows from the definition of a_α, and the fact that the numbers in $\lambda + \delta$ (and $\nu + \delta$) are strictly decreasing.

To prove the result for $N \geqslant l(\lambda)$, notice that the right hand side of the equation remains the same for all $N \geqslant l(\lambda)$, and we already proved the case $N \geqslant n$. □

For a box (i, j) in λ, define the **hook length** $h(i, j)$ by the number of boxes in its row and to its right, or in its column and below it, including itself. Define $f^\lambda = \prod_{(i,j)\in\lambda} h(i, j)$.

THM 4.12 hook length formula

$$\dim V_\lambda = \frac{n!}{f^\lambda}. \tag{47}$$

Proof. Let $\mu = (1^n)$. By Proposition 4.11, denoting $N = l(\lambda)$,

$$\begin{aligned}
\dim V_\lambda &= \chi_{V_\lambda} C(\mu) \\
&= \left[\prod_{i=1}^N x_i^{\lambda_i+N-i}\right]\left(\prod_{1\leqslant i<j\leqslant N}\left(x_i - x_j\right)(x_1+\cdots+x_N)^n\right) \\
&= \left[\prod_{i=1}^N x_i^{\lambda_i+N-i}\right]\left(\sum_{\sigma\in\mathfrak{S}_N}(-1)^\sigma\prod_{i=1}^N x_i^{N-\sigma(i)}(x_1+\cdots+x_N)^n\right) \\
&= \sum_{\substack{\sigma\in\mathfrak{S}_N\\ \sigma(i)\geqslant N-l_i}}(-1)^\sigma\frac{n!}{\prod_{i=1}^N(l_1-N+\sigma(i))!} \quad \text{where } l_i = \lambda_i + N - i \qquad (48) \\
&= \frac{n!}{\prod_{i=1}^N l_i!}\sum_{\sigma\in\mathfrak{S}_N}(-1)^\sigma l_i(l_i-1)\cdots(l_i-N+\sigma(i)+1) \\
&= \frac{n!}{\prod l_i!}\det(l_i\cdots(l_i-N+j+1)) \\
&= \frac{n!}{\prod l_i!}\det\left(l_i^{N-j}\right) = \frac{n!}{\prod l_i!}\prod_{1\leqslant i<j<N}\left(l_i-l_j\right).
\end{aligned}$$

On the other hand, by a combinatorial argument,

$$\prod_{j=1}^{\lambda_1} h(1,j) = \prod_{\substack{1\leqslant k\leqslant\lambda_i+N-i\\ k\neq\lambda_i-\lambda_j+j-1,\forall j}} k. \tag{49}$$

Therefore the two sides are equal. □

We also give a result for representations of $\mathfrak{A}_n$.

THM 4.13 Suppose $\lambda \vdash n$. Then

1. If $\lambda \neq \lambda^\top$, then $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} V_\lambda$ is irreducible.
2. If $\lambda = \lambda^\top$, then $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} V_\lambda = W_\lambda \oplus W'_\lambda$, where $W_\lambda \ncong W'_\lambda$ and both are irreducible representations of $\mathfrak{A}_n$.
3. Any irreducible complex representation of $\mathfrak{A}_n$ appears inside some V_λ and this λ is unique up to a transposition.

Proof.

LEM 4.14 $\mathrm{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n} 1 \cong 1 \oplus \mathrm{sign}$.

By Frobenius reciprocity, $\mathrm{Hom}_{\mathfrak{A}_n}\left(\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} V_\lambda, \mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} V_\lambda\right) = \mathrm{Hom}_{\mathfrak{S}_\lambda}\left(V_\lambda, \mathrm{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n} V_\lambda\right)$.

$$\mathrm{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}V_\lambda \cong V_\lambda \otimes \mathrm{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}1 \cong V_\lambda \oplus (V_\lambda \otimes \mathrm{sign}) \cong V_\lambda \oplus V_{\lambda^\top}. \tag{50}$$

Therefore, if $\lambda \neq \lambda^\top$, then $\mathrm{Hom}_{\mathfrak{S}_n}\left(V_\lambda, \mathrm{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}V_\lambda\right) = \mathbb{C}$, so $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}V_\lambda$ is irreducible. If $\lambda = \lambda^\top$, then $\mathrm{Hom}_{\mathfrak{S}_n}\left(V_\lambda, \mathrm{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}V_\lambda\right) = \mathbb{C}^2$, so $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}V_\lambda$ is the direct sum of two irreducible representations $W_\lambda \oplus W_\lambda'$, and $W_\lambda' \ncong W_\lambda$. □

4.4 Representations of $\mathrm{GL}(V)$

4.4.1 Schur-Weyl Duality

THM 4.15 double centralizer theorem

Let E be a finite dimensional vector space over k, $A, B \subseteq \mathrm{End}(E)$ are two subalgebras, where A is semisimple and $B = \mathrm{End}_A(E)$. Then

1. $A = \mathrm{End}_B(E)$;
2. B is semisimple; and
3. as a representation of $A \otimes B$, $E \cong \bigoplus_i V_i \otimes W_i$, where V_i and W_i are enumerations of irreducible representations of A and B respectively. In particular, there is a bijection between irreducible representations of A and irreducible representations of B.

Proof. Since A is semisimple, $A = \bigoplus_i \mathrm{End}(V_i)$, and $E = \bigoplus_i V_i \otimes W_i$ where $W_i = \mathrm{Hom}_A(V_i, E)$ by the isotypical decomposition. Then $B = \mathrm{End}_A(E) = \mathrm{Hom}_A\left(\bigoplus_i V_i \otimes W_i, \bigoplus_j V_j \otimes W_j\right) = \bigoplus_i \mathrm{End}(W_i)$, so B is semisimple, so similarly $A = \mathrm{End}_B(E)$. □

Consider the case where V is a finite dimensional vector space over $\mathbb{C}$ and $E = V^{\otimes n}$. Let $\mathfrak{S}_n$ act on E by permutation of tensor components, and A be the image of $\mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}(E)$ under this action. Let $\mathfrak{gl}(V)$ be the Lie algebra $(\mathrm{End}(V), [\cdot,\cdot])$.

THM 4.16 Schur-Weyl duality for $\mathfrak{gl}(V)$

The algebra $B = \mathrm{End}_A(E) = S^n\,\mathrm{End}(V)$ is the image of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(V))$ under the natural action on E. That is, B is generated by

$$\Delta_n(b) = b \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes b \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes b \tag{51}$$

for $b \in \mathfrak{gl}(V)$ as an algebra.

THM 4.17 Schur-Weyl duality for $\mathrm{GL}(V)$

The image of $\mathrm{GL}(V)$ in $\mathrm{End}(E)$ spans $B = S^n\,\mathrm{End}(V)$ as a vector space.

COR 4.18 Schur-Weyl duality

As a representation of $\mathfrak{S}_n \times \mathrm{GL}(V)$,

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda, \tag{52}$$

where $L_\lambda = \mathrm{Hom}_{\mathfrak{S}_n}(V_\lambda, V^{\otimes n})$ is either an irreducible representation of $\mathrm{GL}(V)$ or zero, and the non-zero L_λ's are distinct.

For example, if $\lambda = (n)$, then $V_\lambda = 1$, and $L_\lambda = (V^{\otimes n})^{\mathfrak{S}_n} = S^n V$; if $\lambda = (1^n)$, then $L_\lambda = \Lambda^n V$.

We now want to calculate the character of L_λ. Suppose $N = \dim V$, $e_1, \cdots, e_N$ is a basis of V, $g \in \mathrm{GL}(V)$ and $\xi_1, \cdots, \xi_N$ are eigenvalues of g. Suppose $s \in C(\mu) \subseteq \mathfrak{S}_n$, and μ has m_1 1's, m_2 2's, .., m_k k's. Then we can decompose s into cycles of length $m_1, \cdots, m_k$. For a cycle $s_0 = (12\cdots m)$, $(g^{\otimes m} s_0)\left(e_{i_1} \otimes \cdots \otimes e_{i_m}\right) = ge_{i_m} \otimes ge_{i_1} \otimes \cdots \otimes ge_{i_{m-1}}$; therefore, the trace of $g^{\otimes m} s_0$ on $V^{\otimes m}$ is

$$\begin{aligned} \operatorname{tr}_{V^{\otimes m}} &= \sum_{1 \leqslant i_1, \cdots, i_m \leqslant N} g_{i_m i_1} g_{i_1 i_2} \cdots g_{i_{m-1} i_m} \\ &= \operatorname{tr}_V g^m \\ &= p_m(\xi_1, \cdots, \xi_N). \end{aligned} \tag{53}$$

Therefore, for $s \in C(\mu)$, $\operatorname{tr}_{V^{\otimes n}}(g^{\otimes n} s) = p_\mu(\xi_1, \cdots, \xi_N)$. Applying the decomposition is Corollary 4.18,

$$p_\mu(\xi_1, \cdots \xi_N) = \sum_{\lambda \vdash n} \chi_{V_\lambda}(C(\mu)) \chi_{L_\lambda}(g). \tag{54}$$

On the other hand, because the Frobenius character of V_λ is $\sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_{V_\lambda}(C(\mu)) p_\mu = s_\lambda$, we also have

$$p_\mu = \sum_{\lambda \vdash n} \chi_{V_\lambda}(C(\mu)) s_\lambda. \tag{55}$$

Therefore, $\chi_{L_\lambda}(g) = s_\lambda(\xi_1, \cdots, \xi_N)$. We thus obtained

THM 4.19 Weyl character formula

If $l(\lambda) > N$, then $L_\lambda = 0$.

If $l(\lambda) < N$, then $\chi_{L_\lambda}(g) = s_\lambda(\xi_1, \cdots, \xi_N)$.

Using the formula

$$s_\lambda(x_1, \cdots, x_N) = \frac{\det\left(x_i^{\lambda_j + N - j}\right)_{1 \leqslant i,j \leqslant N}}{\det\left(x_i^{N-j}\right)_{1 \leqslant i,j \leqslant N}}, \tag{56}$$

taking $g = \mathrm{id}$, we obtain[2]

COR 4.20 Weyl dimension formula

$$\dim L_\lambda = \prod_{1 \leqslant i < j \leqslant N} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{57}$$

COR 4.21

Let $\lambda + (1^N)$ denote the partition of $n + N$ that adds to 1 to every λ_i.

Then $L_{\lambda + (1^N)} \cong L_\lambda \otimes \Lambda^N V$.

This corollary allows us to generalize L_λ to any $\lambda \in \mathbb{Z}^N$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_N$, not necessarily non-negative. Take $r \geqslant 0$ such that $\lambda + (r^N)$ is indeed a partition (of $n + rN$). We then let

[2]The denominator is zero in this case, so you need to approximate it by $s_\lambda(1, z, \cdots, z^{N-1})$ and take the limit as $z \to 1$.

$$L_\lambda := L_{\lambda+(r^N)} \otimes \left(\Lambda^N V^*\right)^r. \tag{58}$$

The representation $L_{1^N} \cong \Lambda^N V$ is called the **determinant representation**, denoted det. Its dual representation is $\Lambda^N V^* = \det^{-1}$, so $\Lambda^N V \otimes \Lambda^N V^* \cong \mathbb{C}$. Therefore, together with the corollary, this definition is independent on the choice of r.

4.4.2 Algebraic Representations of $\mathrm{GL}(V)$

Denote $G = \mathrm{GL}(V)$. Define R to be the ring of polynomial functions on G, i.e. $R = \mathbb{C}\left[g_{ij}\right]\left[\frac{1}{\det}\right]$. Here g_{ij} are viewed as formal variables, and det is viewed as a polynomial in g_{ij}. For example, if $V \cong \mathbb{C}$, then $G = \mathbb{C}^\times$ and $R = \mathbb{C}\left[x, x^{-1}\right]$.

If we view G as the subvariety of $U = \mathrm{Mat}_{N\times N} \times \mathbb{C}$ defined by the equation $y \cdot \det g = 1$ (where $(g, y) \in \mathrm{Mat}_{N\times N} \times \mathbb{C}$), then the function ring on G is exactly $\mathbb{C}[\mathrm{Mat}_{N\times N}][y]/(y \det - 1) = \mathbb{C}\left[g_{ij}\right]\left[\frac{1}{\det}\right]$.

A finite dimensional representation Y of $\mathrm{GL}(V)$ is called **algebraic** (or **rational** or **polynomial**), if the matrix coefficients of the action on Y belongs to R; i.e., for all $g \in \mathrm{GL}(V)$, $v \in Y$ and $u \in Y^*$, $\langle u, gv\rangle \in R$.

By definition, any subrepresentation or quotient representation of an algebraic representation is also algebraic. V itself is algebraic, and so are V^* and $V^{\otimes n}$. Therefore, L_λ are algebraic.

Denote $\Lambda^+ = \left\{(\lambda_1 \geqslant \cdots \geqslant \lambda_N) \in \mathbb{Z}^N\right\}$.[3]

THM 4.22 Peter-Weyl

1. Let $G \times G$ acts on R as $((g, h)(\varphi))(x) = \varphi\left(g^{-1} x h\right)$. Then as a $G \times G$-representation, $R \cong \bigoplus_{\lambda\in\Lambda^+} L_\lambda^* \otimes L_\lambda$.

2. Any algebraic representation of G is completely reducible.

Proof. $G \times G$ acts on $\mathrm{Mat}_{N\times N}$ by $(g, h)\varphi = g\varphi h^{-1}$. Then as a $G \times G$-representation, $\mathrm{Mat}_{N\times N} \cong V \otimes V^*$, where the first G acts on V and the second G acts on V^*. $\mathbb{C}[\mathrm{Mat}_{N\times N}] = \mathrm{Sym}(V^* \otimes V) = \bigoplus_{n\geqslant 0} S^n(V^* \otimes V)$, and

$$(V^* \otimes V)^{\otimes n} = \left(\bigoplus_{\substack{\lambda\vdash n \\ l(\lambda)\leqslant N}} L_\lambda^* \otimes V_\lambda^*\right) \otimes \left(\bigoplus_{\substack{\mu\vdash n \\ l(\mu)\leqslant N}} L_\mu \otimes V_\mu\right), \tag{59}$$

so

$$S^n(V^* \otimes V) = \mathrm{Hom}_{\mathfrak{S}_n}\left(\mathbb{C}, (V^* \otimes V)^{\otimes n}\right) = \bigoplus_{\substack{\lambda\vdash n \\ l(\lambda)\leqslant N}} L_\lambda^* \otimes L_\lambda \tag{60}$$

by Schur's lemma.

[3] the *dominant integral weights*

Let ρ be the representation of $G\times G$ on $\mathbb{C}^\times$ by $(g,h)\mapsto(\det g)^{-1}(\det h)$, or as a $G\times G$-representation, $\rho\cong\Lambda^N V^*\otimes\Lambda^N V$. Extend the $G\times G$-action on $\mathbb{C}[\mathrm{Mat}_{N\times N}]$ to $\mathbb{C}[\mathrm{Mat}_{N\times N}][y]$ by requiring that $G\times G$ acts on y via ρ^{-1}. Then

$$(g,h)\cdot(y\det -1)=((g,h)\cdot y)\big((\det g)^{-1}(\det h)\cdot\det\big)-1=y\det -1, \tag{61}$$

so

$$\begin{aligned}\mathbb{C}[\mathrm{Mat}_{N\times N}][y]&=\bigoplus_{d\geqslant 0}\bigoplus_{\substack{\lambda\vdash n\\ l(\lambda)\leqslant N}}(L_\lambda^*\otimes L_\lambda)y^d\\ &=\bigoplus_{d\geqslant 0}\bigoplus_{\substack{\lambda\vdash n\\ l(\lambda)\leqslant N}}(L_\lambda^*\otimes L_\lambda)\otimes\rho^{-d}\\ &=\bigoplus_{d\geqslant 0}\bigoplus_{\substack{\lambda\vdash n\\ l(\lambda)\leqslant N}}(L_\lambda^*\otimes L_\lambda)\otimes\left(\Lambda^N V\otimes\Lambda^N V^*\right)^{\otimes d}\\ &=\bigoplus_{d\geqslant 0}\bigoplus_{\substack{\lambda\vdash n\\ l(\lambda)\leqslant N}}L^*_{\lambda-(d^N)}\otimes L_{\lambda-(d^N)}\end{aligned} \tag{62}$$

So far we have copies of the same L_λ^* appearing from various d. Recall that R is the quotient of $\mathbb{C}[\mathrm{Mat}_{N\times N}][y]$ by $(y\det -1)$. In this quotient, $(L_\lambda^*\otimes L_\lambda)y^d$ is identified with $\left(L^*_{\lambda+(1^N)}\otimes L_{\lambda+(1^N)}\right)y^{d+1}$. Therefore, as $G\times G$-representations,

$$R\cong\bigoplus_{\lambda\in\Lambda^+}L_\lambda^*\otimes L_\lambda. \tag{63}$$

□

COR 4.23 $\{L_\mu:\mu\in\Lambda^+\}$ is a complete list of pairwise non-isomorphic irreducible algebraic representations of $\mathrm{GL}(V)$.

4.5 The Okounkov-Vershik Approach

We treat $\mathfrak{S}_k$ as the subgroup of $\mathfrak{S}_{k+1}$ fixing $k+1$.

Suppose A and B are semisimple algebras, $\tau:B\to A$ an algebra homomorphism. Since A and B are semisimple, we can write them as $A=\bigoplus_{V\in\mathrm{Irr}(A)}\mathrm{End}(V)$, $B=\bigoplus_{U\in\mathrm{Irr}(B)}\mathrm{End}(U)$. For $U\in\mathrm{Irr}(B)$ and $V\in\mathrm{Irr}(A)$, let $M_{V,U}=\mathrm{Hom}_B(U,V)$, where V is seen as an representation of B by composition $B\xrightarrow{\tau}A\xrightarrow{\rho_V}\mathrm{End}(V)$. Then $V\cong\bigoplus_{U\in\mathrm{Irr}(B)}U\otimes M_{V,U}$ as representations of B ($M_{V,U}$ records the multiplicity of U and does not have a B-action).

Define the **centralizer** $Z_B(A)=\{a\in A:a\tau(b)=\tau(b)a,\forall b\in B\}$. The centralizer is a subalgebra of A, and acts on $M_{V,U}$ by $(a\cdot\varphi)(u)=a(\varphi(u))$.

Since A can be written as $\bigoplus_{V\in\mathrm{Irr}(A)}\mathrm{End}(V)$, we can write $Z_B(A)$ as

$$
\begin{aligned}
Z_B(A) &= Z_B\Big(\bigoplus_{V\in \mathrm{Irr}(A)} \mathrm{End}(V)\Big) \\
&= \bigoplus_{V\in \mathrm{Irr}(A)} \mathrm{Hom}_B(V,V) \\
&= \bigoplus_{V\in \mathrm{Irr}(A)} \mathrm{Hom}_B\Big(\bigoplus_{U\in \mathrm{Irr}(B)} U\otimes M_{V,U}, \bigoplus_{U'\in \mathrm{Irr}(B)} U\otimes M_{V,U'}\Big) \\
&\overset{\text{Schur lemma}}{=\!=\!=} \bigoplus_{\substack{V\in \mathrm{Irr}(A)\\ U\in \mathrm{Irr}(B)\\ M_{V,U}\neq 0}} \mathrm{End}(M_{V,U}).
\end{aligned} \tag{64}
$$

As a corollary, $Z_B(A)$ is commutative if and only if for all $U \in \mathrm{Irr}(B)$ and $V \in \mathrm{Irr}(A)$, $\dim \mathrm{Hom}_B(U,V) \leqslant 1$.

Suppose $H \subseteq G$ are finite groups. If $\tau : B \to A$ is an inclusion of group algebras $\mathbb{C}[H] \hookrightarrow \mathbb{C}[G]$, then $Z_{\mathbb{C}[H]}(\mathbb{C}[G])$ consists of elements $\sum_{g\in G} a_g g$ where $a_g = a_{hgh^{-1}}$ for all $h \in H$. Hence, $Z_{\mathbb{C}[H]}(\mathbb{C}[G])$ has a basis $b_C = \sum_{g\in C} g$ where C is an H-conjugacy class in G.

We now take $\tau : B \to A$ to be the inclusion $\mathbb{C}[\mathfrak{S}_m] \hookrightarrow \mathbb{C}[\mathfrak{S}_n]$ for $0 \leqslant m \leqslant n$ and denote $Z_m(n) = Z_{\mathbb{C}[\mathfrak{S}_m]}(\mathbb{C}[\mathfrak{S}_n])$. The $\mathfrak{S}_m$-conjugacy classes in $\mathfrak{S}_n$ are cycles with marked elements $m+1, m+2, \cdots, n$. For example, when $m = 3$, $n = 6$, some marked cycles are

$$
(* * 4)(5)(6), \quad (* * 5)(4\,6), \quad (* * 5)(* 4)(6), \quad \cdots \tag{65}
$$

For another example, if $m = n-1$, then $(* \, n)$ is a conjugacy class, and the corresponding basis element $b_{(*n)} = \sum_{j=1}^{n-1} (j\; n)$ is called the **n-th Jucys-Murphy element**.

Let $\mathfrak{S}_{[m+1,n]} = \{g \in \mathfrak{S}_n : g(i) = i, \forall 1 \leqslant i \leqslant m\}$. $Z_m(n)$ contains the following elements:

- center of $\mathbb{C}[\mathfrak{S}_m]$, which is $Z_m(m)$;
- $\mathbb{C}[\mathfrak{S}_{[m+1,n]}]$;
- $J_{m+1}, J_{m+2}, \cdots, J_n$; they mutually commute.

THM 4.24 The algebra $Z_m(n)$ is generated by the above elements.

As a corollary, $Z_{n-1}(n)$ is commutative, so any irreducible representation of $\mathfrak{S}_{n-1}$ appears at most once in the restriction of any irreducible representation of $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$, and according to Schur lemma, J_n acts by scalar multiplication on each irreducible $\mathfrak{S}_{n-1}$-subrepresentation of any irreducible $\mathfrak{S}_n$-representation. (This is a consequence of the *branching law*: the restriction of V_λ from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$ decomposes into $V_{\lambda\setminus\square}$ where $\square$ ranges through all removable box of λ.)

Based on this, we define the **branching graph** as an infinite directed tree with vertices isomorphism classes of irreducible representations of symmetric groups, and edges $U \to V$ where $V \in \mathrm{Irr}(\mathfrak{S}_n)$, $U \in \mathrm{Irr}(\mathfrak{S}_{n-1})$ for some n, such

$$
\begin{tikzcd}
\mathfrak{S}_1 & & & \mathrm{triv}_1 \arrow[dl, "1"'] \arrow[dr, "-1"] & & \\
\mathfrak{S}_2 & & \mathrm{triv}_2 \arrow[d, "2"] \arrow[dr, "-1"] & & \mathrm{sign}_2 \arrow[dl, "1"'] \arrow[d, "-2"] & \\
\mathfrak{S}_3 & & \mathrm{triv}_3 \arrow[dl, "3"'] \arrow[d, "-1"] & \mathrm{std}_3 \arrow[dl, "2"'] \arrow[d, "0"] \arrow[dr, "-2"] & \mathrm{sign}_3 \arrow[d, "1"] \arrow[dr, "-3"] & \\
\mathfrak{S}_4 & \mathrm{triv}_4 & \mathrm{std}_4 & \mathbb{C}^2 & \mathrm{std}_4 \otimes \mathrm{sign}_4 & \mathrm{sign}_4
\end{tikzcd}
$$

$\vdots$

that U appears in $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(V)$. This graph remains invariant if we tensor every representation with the sign representation, replacing U by $U \otimes \mathrm{sign}_n$; this makes it "symmetric".

Suppose $V^n \in \mathrm{Irr}(\mathfrak{S}_n)$ for $n \in \mathbb{Z}_+$, and denote $\mathrm{Path}(V^m, V^n)$ the set of all paths from V^m to V^n in the branching graph. Let $\mathrm{Path}(V^n) = \mathrm{Path}(\mathrm{triv}_1, V^n)$ and $\mathrm{Path}_n = \bigsqcup_{V^n \in \mathrm{Irr}(\mathfrak{S}_n)} \mathrm{Path}(V^n)$. For $P = (V^m \to \cdots \to V^n) \in \mathrm{Path}(V^m, V^n)$, write $V^m(P)$ for the copy of V^m inside V^n by the composition of inclusions corresponding to P. Then, as $\mathfrak{S}_m$-representations,

$$V^n = \bigoplus_{V^m \in \mathrm{Irr}(\mathfrak{S}_m)} \bigoplus_{P \in \mathrm{Path}(V^m, V^n)} V^m(P). \tag{66}$$

Let $\varphi_P : V^m \hookrightarrow V^n$ be the embedding corresponding to path P; they form a basis for $\mathrm{Hom}_{\mathfrak{S}_m}(V^m, V^n)$.

$Z_m(n)$ acts on $\mathrm{Hom}_{\mathfrak{S}_m}(V^m, V^n)$ by $(a \cdot \varphi)(u) = a \cdot \varphi(u)$. Since $J_k \in Z_m(n)$, J_k also acts on $\mathrm{Hom}_{\mathfrak{S}_m}(V^m, V^n)$. For every edge $V^{k-1} \to V^k$ in the path, assign a number w_k to it, defined as the scalar by which J_k acts on V^{k-1}: $J_k|_{V^{k-1}} = w_k \,\mathrm{id}_{V^{k-1}}$. Then J_k acts on φ_P as $J_k \cdot \varphi_P = w_k \varphi_P$. The weight vector $(w_{m+1}, \cdots, w_n)$ is denoted by w_P.

LEM 4.25 $\{\varphi_P : P \in \mathrm{Path}(V^m, V^n)\}$ is a basis for $\mathrm{Hom}_{\mathfrak{S}_m}(V^m, V^n)$.

Since $V^1 = \mathrm{triv}_1$, $\mathrm{Hom}_{\mathfrak{S}_n}\left(V^1, V^n\right) = V^n$. Let $v_P = \varphi_P$ for $P \in \mathrm{Path}(V^n)$. As a corollary, $\{v_P : P \in \mathrm{Path}(V^n)\}$ is a basis for V^n, and for any $1 \leqslant k \leqslant n$, $J_k v_P = w_k v_P$.

As another corollary, if $P \in \mathrm{Path}(V^m)$ and $Q \in \mathrm{Path}(V^m, V^n)$, then v_{PQ} is proportional to $\varphi_Q(v_P)$.

THM 4.26 Suppose $P, P' \in \mathrm{Path}_n$. Then $w_P = w_{P'}$ if and only if $P = P'$.

Denote $\mathrm{wt}_n = \{w_P : P \in \mathrm{Path}_n\}$. We say w_P and $w_{P'}$ are **r-equivalent** if they are the weights of two paths leading to the same irreducible representation. This is an equivalence relation, therefore the study of $\mathrm{Irr}(\mathfrak{S}_n)$ reduces to the study of equivalence classes of r-equivalence.

Let $\mathrm{Path}(P,i)$ be the set of paths that differs from P *only* at level i. Denote $V_{P,i} = \mathrm{Span}_{\mathbb{C}}\{v_P : P \in \mathrm{Path}(P,i)\} \subseteq V^n$.

PROP 4.27 $V_{P,i} \subseteq V^n$ is a $Z_{i-1}(i+1)$-submodule, and it's an irreducible $Z_{i-1}(i+1)$-module.

Recall that $Z_{i-1}(i+1)$ is generated by $Z_{i-1}(i-1)$, $X_1 = J_i$, $X_2 = J_{i+1}$ and $T = (i\ i+1)$. Observe that X_1, X_2 and T has the following relations:

$$X_1X_2 = X_2X_1,\ T^2 = 1,\ TX_1 = X_2T - 1. \tag{67}$$

The algebra generated by the above three relations is called a **degenerate affine Hecke algebra** $\mathcal{H}(2)$. Then there is an algebra homomorphism $\mathcal{H}(2) \to Z_{i-1}(i+1)$.

PROP 4.28 If M is an irreducible $Z_{i-1}(i+1)$-module, then M is also irreducible as an $\mathcal{H}(2)$ -module.

THM 4.29 The finite-dimensional irreducible representations of $\mathcal{H}(2)$ are classified by $(a,b) \in \mathbb{C}^2 \mapsto L(a,b)$, where (a,b) are simultaneous eigenvalues of X_1 and X_2 in $L(a,b)$:

1. If $b = a+1$, then $L(a,b) = \mathbb{C}$, $T \mapsto 1$, $X_1 \mapsto a$ and $X_2 \mapsto b$.
2. If $b = a-1$, then $L(a,b) = \mathbb{C}$, $T \mapsto -1$, $X_1 \mapsto a$ and $X_2 \mapsto b$.
3. If $b \neq a \pm 1$, then $L(a,b) = \mathbb{C}^2$, $T \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $X_1 \mapsto \begin{pmatrix} a & -1 \\ & b \end{pmatrix}$ and $X_2 \mapsto \begin{pmatrix} b & 1 \\ & a \end{pmatrix}$.
4. X_1 and X_2 are diagonalizable in $L(a,b)$ if and only if $a \neq b$.

$V_{P,i}$ is isomorphic to $\mathrm{Hom}_{\mathfrak{S}_{i-1}}(V^{i-1}, V^{i+1})$ as a $Z_{i-1}(i+1)$-module, and this is an irreducible representation for $Z_{i-1}(i+1)$. Since there is an algebra homomorphism from $\mathcal{H}(2)$ to $Z_{i-1}(i+1)$, $V_{P,i}$ is also an irreducible $\mathcal{H}(2)$-module. Therefore, $V_{P,i}$ is isomorphic to one of the L defined above. By the classification, we obtain

THM 4.30 **$\mathrm{Path}(P,i)$ theorem** Let $w_P = (w_1, \cdots, w_n) \in \mathrm{wt}_n$. Then

1. $w_i \neq w_{i+1}$;
2. If $w_{i+1} = w_i \pm 1$, then $\mathrm{Path}(P,i) = \{P\}$;
3. If $w_{i+1} \neq w_i \pm 1$, then $\mathrm{Path}(P,i) = \{P, P'\}$ and
$$w_{P'} = (w_1, \cdots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \cdots, w_n); \tag{68}$$
4. If $i < n-1$, $w_i = w_{i+1} \pm 1$ implies that $w_{i+2} \neq w_i$.

According to this theorem, we define an **admissible transposition** on $\mathbb{C}^n$ as a transposition $(w_1, \cdots, w_n) \mapsto (\cdots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \cdots)$ where $w_{i+1} \neq w_i \pm 1$. We say two elements of $\mathbb{C}^n$ are **c-equivalent**, denoted by $\underset{c}{\sim}$, if one is obtained from the other by a sequence of admissible transpositions. A **combinatorial weight** is an element in $\mathbb{C}^n$ such that every element w c-equivlent to it satisfies $w_1 = 0$, $w_i \neq w_{i+1}$ for all $1 \leqslant i \leqslant n-1$, and $w_{i+1} = w_i \pm 1$ implies $w_{i+2} \neq w_i$ for all $1 \leqslant i \leqslant n-2$. Let cwt_n be the set of combinatorial weights.

COR 4.31

1. $\mathrm{wt}_n \subseteq \mathrm{cwt}_n$, and wt_n is a union of c-equivalence classes;
2. c-equivalence implies r-equivalence in wt_n.

Therefore, the number of r-equivalence classes is no larger than the number of c-equivalence classes in wt_n, and is in turn no larger the that in cwt_n. On the other hand, the number of r-equivalence classes is equal to the number of irreducible representations of $\mathfrak{S}_n$, which is $p(n)$, the partition number of n. We now prove the final combinatorial lemma, that

LEM 4.32 Every c-equivalence class in cwt_n contains an element of the form

$$\begin{array}{l}(\\ \quad 0, 1, 2, \cdots, n_1 - 1, \\ \quad -1, 0, 1 \cdots, n_2 - 2, \\ \quad -2, -1, 0, \cdots, n_3 - 3, \\ \quad \cdots, \\ \quad 1-k, 2-k, 3-k, \cdots, n_k - k \\ )\end{array} \tag{69}$$

for some positive integers $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$ such that $\sum_i n_i = n$.

In particular, the number of c-equivalence classes in cwt_n is no larger than $p(n)$, completing the cycle of inequalities.

Proof. Consider the lexicographic order on an equivalence class in cwt_n. Let $(w_1, \cdots, w_n)$ be the maximal element in its equivalence classes. This element has the desired form. □

This implies $\mathrm{cwt}_n = \mathrm{wt}_n$, c-equivalence is r-equivalence, and the $(n_1, \cdots, n_k)$ in the lemma is unique; this establishes a bijection between irreducible representations of $\mathfrak{S}_n$ with a partition $(n_1, \cdots, n_k)$ of n.

To see the relation with Young diagrams, suppose λ be a partition of n and T is a standard Young tableaux of of shape λ. Its **content** is defined by $c(T) = (x_1 - y_1, \cdots, x_n - y_n) \in \mathbb{Z}^n$, where (x_i, y_i) is the coordinate of the box labelled by i. The content $T \mapsto c(T)$ gives a bijection between standard Young tableaux with n boxes and cwt_n, and the shape of the tableaux is equal to the partition defined in Lemma 4.32. As a corollary, V_λ has a basis given by v_T, where T iterates over all standard Young tableaux of shape λ, and the Jucys-Murphy elements act on this basis as $J_i v_T = (x_i - y_i) v_T$, where (x_i, y_i) is the coordinate of the box labelled i in T. As another corollary, we obtain the branching rule $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(V_\lambda) = \bigoplus_{\mu=\lambda\setminus\square} V_\mu$, and J_n acts on V_μ by scalar multiplication by the content of the removed box.

5

Section 5
Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

Let $G = \mathrm{GL}_2(\mathbb{F}_q)$, with $|G| = (q^2-1)(q^2-q)$. Suppose $A \in G$ with eigenvalues λ_1, λ_2. Let C_A be the conjugacy class of A.

Case a: parabolic, $\lambda_1 = \lambda_2 = x \in \mathbb{F}_q^\times$. If A is diagonalizable, then $|C_A| = 1$. If not, then $A \sim \begin{pmatrix} x & 1 \\ & x \end{pmatrix}$, and $|C_A| = \frac{|G|}{(q-1)q} = q^2 - 1$.

Case b: hyperbolic, $\lambda_1 \neq \lambda_2 \in \mathbb{F}_q^\times$. Then $|C_A| = \frac{|G|}{(q-1)^2} = q^2 + q$.

Case c: elliptic, $\lambda_1 \neq \lambda_2 \notin \mathbb{F}_q$. Consider a quadratic extension of $\mathbb{F}_q$, $\mathbb{F}_{q^2} = \mathbb{F}_q[\sqrt{\varepsilon}]$. Then $\lambda_1 = \alpha_1 + \alpha_2\sqrt{\varepsilon}$, $\lambda_2 = \alpha_1 - \alpha_2\sqrt{\varepsilon} = \overline{\lambda_1}$, where $\alpha_i \in \mathbb{F}_q^*$, and $A \sim \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$. $|C_A| = q^2 - q$.

Representatives A	$\lvert C_A \rvert$	number of classes
$\begin{pmatrix} x & \\ & x \end{pmatrix}$	1	$q-1$
$\begin{pmatrix} x & 1 \\ & x \end{pmatrix}$	q^2-1	$q-1$
$\begin{pmatrix} x & \\ & y \end{pmatrix}, x \neq y$	q^2+q	$\frac{1}{2}(q-1)(q-2)$
$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$	q^2-q	$\frac{1}{2}q(q-1)$

We start with 1-dimensional representations. Any 1-dimensional representation factors through $G/[G,G]$.

LEM 5.1 $[G,G] = \mathrm{SL}_2(\mathbb{F}_q)$; the quotient map $G \to G/[G,G]$ is given by $g \mapsto \det g$.

Therefore, $G/[G,G] \cong \mathbb{F}_q^* \cong C_{q-1}$. Therefore, any 1-dimensional representation of G is of the form $\rho(g) = \xi(\det g)$, where ξ is a 1-dimensional representation of $\mathbb{F}_q^*$. This representation will be denoted $\mathbb{C}_\lambda$. This gives us $q-1$ 1 -dimensional representations.

We can also construct the **principal series representations**. Let B be the subgroup of upper diagonal matrices (the **Borel subgroup**), U the subgroup of upper diagonal matrices with diagonals equal to 1, and T the subgroup of diagonal matrices. Then $B / [B,B] = B / U = T$, and if λ_i are representations of $\mathbb{F}_q^\times$, then $\begin{pmatrix} a & \\ & d \end{pmatrix} \mapsto \lambda_1(a)\lambda_2(b)$ gives a representation of T. The composition $\rho_{\lambda_1\lambda_2} : B \to T \to \mathbb{C}^\times$ gives a representation of B, denoted $\mathbb{C}_{\lambda_1\lambda_2}$, and its induces a representation of G, denoted $V_{\lambda_1\lambda_2} = \mathrm{Ind}_B^G \mathbb{C}_{\lambda_1\lambda_2}$.

THM 5.2

- If $\lambda_1 \neq \lambda_2$, then $V_{\lambda_1\lambda_2}$ is irreducible. Moreover, $V_{\lambda_1\lambda_2}$ are distinct for different unordered pairs $\{\lambda_1,\lambda_2\}$ $(\lambda_1 \neq \lambda_2)$.
- If $\lambda_1 = \lambda_2 = \mu$, then $V_{\mu\mu} \cong \mathbb{C}_\mu \oplus W_\mu$, where W_μ is an irreducible representation of G. Moreover, W_μ are distinct for different μ's.

Proof. By $G = B \cup BsB$ where $s = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, and $sBs^{-1} \cap B = T$. By Theorem 3.17, $\mathrm{Res}_B^G\left(V_{\lambda_1\lambda_2}\right) = \rho_{\lambda_1\lambda_2} \oplus \mathrm{Ind}_T^B\left(\rho^s_{\lambda_1\lambda_2}\right)$, where $\rho^s_{\lambda_1\lambda_2} : \begin{pmatrix} a & \\ & d \end{pmatrix} \mapsto \rho_{\lambda_1\lambda_2}\left(s^{-1}\begin{pmatrix} a & \\ & d \end{pmatrix}s\right) = \lambda_2(a)\lambda_1(d)$. When $\lambda_1 \neq \lambda_2$, $\rho^s_{\lambda_1,\lambda_2}$ and $\mathrm{Res}_T^B\left(\rho_{\lambda_1\lambda_2}\right)$ are then disjoint, so $V_{\lambda_1\lambda_2}$ are irreducible. Moreover, if $\lambda_1 \neq \lambda_2$, $\lambda_1' \neq \lambda_2'$,

$$
\begin{aligned}
\mathrm{Hom}_G\left(V_{\lambda_1\lambda_2}, V_{\lambda_1',\lambda_2'}\right) &\cong \mathrm{Hom}_B\left(\rho_{\lambda_1\lambda_2} \oplus \mathrm{Ind}_T^B\left(\rho^s_{\lambda_1\lambda_2}\right), \rho_{\lambda_1'\lambda_2'}\right) \\
&= \mathrm{Hom}_B\left(\rho_{\lambda_1\lambda_2}, \rho_{\lambda_1'\lambda_2'}\right) \oplus \mathrm{Hom}_T\left(\rho^s_{\lambda_1\lambda_2}, \mathrm{Res}\,\rho_{\lambda_1'\lambda_2'}\right) \\
&= \mathrm{Hom}_B\left(\rho_{\lambda_1\lambda_2}, \rho_{\lambda_1'\lambda_2'}\right) \oplus \mathrm{Hom}_T\left(\mathrm{Res}\,\rho_{\lambda_2\lambda_1}, \mathrm{Res}\,\rho_{\lambda_1'\lambda_2'}\right) \\
&= \mathbb{C} \text{ iff } \{\lambda_1,\lambda_2\} = \{\lambda_1',\lambda_2'\}.
\end{aligned} \tag{70}
$$

When $\lambda_1 = \lambda_2 = \mu$, $\mathrm{Res}_B^G\left(\mathbb{C}_\mu\right) = \rho_{\mu\mu}$, so $\mathrm{Hom}_G\left(C_\mu, V_{\mu\mu}\right) \cong \mathrm{Hom}_B\left(\mathrm{Res}_B^G\left(\mathbb{C}_\mu\right), \rho_{\mu\mu}\right) \cong \mathbb{C}$, so $\mathbb{C}_\mu \subseteq V_{\mu\mu}$. Therefore, $V_\mu\mu = C_\mu \oplus W_\mu$. Since $\mathrm{Hom}_G\left(V_{\mu\mu}, V_{\mu\mu}\right) = \mathbb{C}^2$ by the same calculation as Equation 70, we deduce $\mathrm{Hom}_G\left(W_\mu, W_\mu\right) = \mathbb{C}$, so W_μ is irreducible of dimension q. Moreover, $\mathrm{Hom}_G\left(V_{\mu\mu}, V_{\nu\nu}\right) = \mathrm{Hom}_B\left(\rho_{\mu\mu}, \rho_{\nu\nu}\right) \oplus \mathrm{Hom}_T\left(\mathrm{Res}\,\rho_{\mu\mu}, \mathrm{Res}\,\rho_{\nu\nu}\right) = \mathbb{C}^2$ iff $\mu = \nu$, so $W_\mu \cong W_\nu$ iff $\mu = \nu$. ☐

Therefore $V_{\lambda_1\lambda_2}$ gives us $\frac{1}{2}(q-1)(q-2)$ many $(q+1)$-dimensional irreducible representations, and W_μ gives us $q-1$ many q-dimensional irreducible representations. By Proposition 4.11,

Conjugacy class	$\chi_{V_{\lambda_1\lambda_2}}(g)$
$\begin{pmatrix} x & \\ & x \end{pmatrix}$	$(q+1)\lambda_1(x)\lambda_2(x)$
$\begin{pmatrix} x & 1 \\ & x \end{pmatrix}$	$\lambda_1(x)\lambda_2(x)$
$\begin{pmatrix} x & \\ & y \end{pmatrix}, x \neq y$	$\lambda_1(x)\lambda_2(y) + \lambda_2(x)\lambda_1(y)$

$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$	0

Another family of representations are the **complementary series representations.** We can identify G with the group of invertible $\mathbb{F}_q$-linear automorphisms of $\mathbb{F}_{q^2}$. This way, G contains the cyclic subgroup $\mathbb{F}_{q^2}^\times = \left\{x + y\sqrt{\varepsilon} : x, y \in \mathbb{F}_q, (x, y) \neq (0, 0)\right\}$, or in $\mathrm{GL}_2\left(\mathbb{F}_q\right)$, $\left\{\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} : (x, y) \neq (0, 0)\right\}$. Denote the subgroup by K. Then every character ν of K can be induced to a representation of G, denoted Y_ν. By Proposition 4.11,

Conjugacy class	$\chi_{Y_\nu}(g)$
$\begin{pmatrix} x & \\ & x \end{pmatrix}$	$q(q-1)\nu(x)$
$\begin{pmatrix} x & 1 \\ & x \end{pmatrix}$	0
$\begin{pmatrix} x & \\ & y \end{pmatrix}, x \neq y$	0
$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$	$\nu\left(x + y\sqrt{\varepsilon}\right) + \nu\left(x - y\sqrt{\varepsilon}\right)$

Notice that $\nu\left(x - y\sqrt{\varepsilon}\right) = \nu\left(x + y\sqrt{\varepsilon}\right)^q$ via the Frobenius automorphism $x \mapsto x^q$. Also via the Frobenius automorphism, $Y_\nu \cong Y_{\nu^q}$.

Consider the virtual representation $X_\nu = W_1 \otimes V_{\alpha 1} - V_{\alpha 1} - Y_\nu$, where $\alpha = \nu|_{\mathbb{F}_q^\times}$.

Conjugacy class	$\chi_{X_\nu}(g)$
$\begin{pmatrix} x & \\ & x \end{pmatrix}$	$(q-1)\alpha(x)$
$\begin{pmatrix} x & 1 \\ & x \end{pmatrix}$	$-\alpha(x)$
$\begin{pmatrix} x & \\ & y \end{pmatrix}, x \neq y$	0
$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$	$-\nu\left(x + y\sqrt{\varepsilon}\right) - \nu\left(x + y\sqrt{\varepsilon}\right)^q$

LEM 5.3 Assume $\nu^q \neq \nu$. Then $\left(\chi_{X_\nu}, \chi_{X_\nu}\right) = 1$ and $\chi_{X_\nu}(1) = q - 1 > 0$, so X_ν is an irreducible representation. $X_\nu \cong X_{\nu'}$ iff $\nu' \neq \nu, \nu^q$.

X_ν thus gives us $\frac{1}{2}q(q-1)$ many $(q-1)$-dimensional irreducible representations.

$V_{\lambda_1\lambda_2}$ $(\lambda_1 \neq \lambda_2)$, W_μ, $\mathbb{C}_\mu$ and X_ν $(\nu^q \neq \nu)$ thus gives all irreducible representations of $\mathrm{GL}_2\left(\mathbb{F}_q\right)$.

6

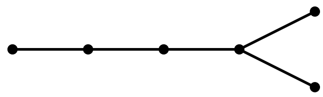
Section 6
Representations of Quivers

Suppose $\Gamma = (I, E)$ is a connected graph with no self loops. Denote its adjacency matrix by R_Γ. Γ is called a **Dynkin diagram** if the quadratic form on $\mathbb{R}^N$ defined by $A_\Gamma = 2I - R_\Gamma$ is positive definite.

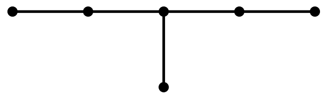
THM 6.1 All Dynkin diagrams are classified by three families:

A_N (shown here A_5)

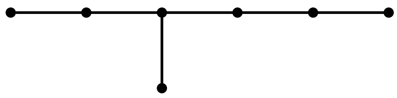
D_N (shown here D_6)



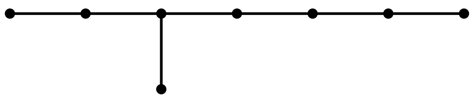
E_6



E_7



E_8



Suppose Γ is a Dynkin diagram, define the bilinear form $B(x, y) = x^\top A_\Gamma y$ on $\mathbb{Z}^n$. Then B is positive definite, and $B(x, x) = 2\sum x_i^2 - \sum_{i \neq j} r_{ij} x_i x_j$, where r_{ij} is 1 or 0 determined by whether i and j are connected or not. In particular, $B(x, x)$ is always even. A **root** with respect to a certain positive definite inner product B is a shortest non-zero vector in $\mathbb{Z}^n$, and in this case, $v \in \mathbb{Z}^n$ such that $B(v, v) =$

2. Obviously there are only finitely many roots. Define the **simple roots** $\alpha_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 on the i-th component.

LEM 6.2 If α is a root, $\alpha = \sum_i k_i \alpha_i$, then either all k_i are non-negative, or all k_i are non-positive. α is called a **positive root** or a **negative root** correspondingly.

Proof. Argue by contradiction. Assume that $k_i > 0$, $k_j < 0$ and $k_s = 0$ for all s in between i and j. Let ε be the edge connecting i with the other vertex i' towards j. Delete ε and Γ breaks into two parts, Γ_1 containing i and Γ_2 containing j. Suppose β and γ are vectors obtained by restricting the indices of α to vertices of Γ_1 and Γ_2, respectively, then $\alpha = \beta + \gamma$, $B(\beta, \beta) \geqslant 2$, $B(\gamma, \gamma) \geqslant 2$, and $2 = B(\alpha, \alpha) = B(\beta, \beta) + B(\gamma, \gamma) + 2B(\beta, \gamma)$. However, $B(\beta, \gamma) = -k_i k_{i'} \geqslant 0$, contradiction. □

Let R be the set of roots, R^+ the set of positive roots, and R^- the set of negative roots ($R^- = -R^+$).

If α is a root, the **reflection** s_α on $\mathbb{R}^n$ is $s_\alpha(v) = v - B(v, \alpha)\alpha$. It is equivalent as the reflection along the hyperplain H_α orthogonal to α. If $\alpha = \alpha_i$ is a simple root, $s_\alpha = s_i$ is called a **simple reflection**, and the subgroup of $\mathrm{O}(\mathbb{R}^n)$ generated by s_i is called the **Weyl group** W. If the number of roots is finite, then W is finite. If we fix a labelling $1, 2, \cdots, n$ of the vertices of Q, then $c = s_1 s_2 \cdots s_n \in W$ is called the **Coxeter element**.

LEM 6.3 Let $\beta = \sum k_i \alpha_i \neq 0$ with $k_i \geqslant 0$ for all i. Then there is some $N > 0$, such that $c^N \beta$ has at least one strictly negative coefficient.

Suppose Q is a quiver, or equivalently a directed graph. For any $i \in I$, define the representation $S(i)$ by assigning to each vertex j the vector space $S(i)_j = 0$ if $j \neq i$ and $\mathbb{C}$ if $j = i$, and for any $h \in E$, the linear map assigned to h is $x_h = 0$. $S(i)$ is an irreducible representation, and are non-isomorphic for different i.

THM 6.4 Assume Q has no oriented cycles. Then $S(i)$ are all irreducible representations of Q.

Proof. Suppose V is a irreducible representation of Q, let $I' = \{i \in I : V_i \neq 0\}$. We want to find an $i \in I'$ such that $\dim V_i \cdot S(i)$ is a subrepresentation of V. A sufficient condition for such an i if that for any $j \in I'$, there is no edge from i to j. This is ensured by the assumption that Q has no oriented cycles. □

For a representation V of Q, define its **dimension vector** by

$$\dim V = (\dim V_1, \cdots, \dim V_n) \in \mathbb{Z}^N_{\geqslant 0}. \tag{71}$$

For $v \in \mathbb{Z}^I_{\geqslant 0}$, let the representation space be $\mathrm{Rep}(Q, v)$ be the space of representations V of Q such that $\dim V = v$, Equivalently,

$$\mathrm{Rep}(Q, v) = \bigoplus_{h \in E} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{v_{h'}}, \mathbb{C}^{v_{h''}}). \tag{72}$$

$\mathrm{GL}(v) := \prod_{i \in I} \mathrm{GL}(v_i)$ acts on $\mathrm{Rep}(Q, v)$ by conjugation: $x_h \mapsto g_{h''} x_h g_{h'}^{-1}$. Two representations in $\mathrm{Rep}(Q, v)$ are isomorphic if and only if they are in the same $\mathrm{GL}(v)$-orbit.

A quiver Q is of **finite type** if the number of *indecomposable* representations is finite up to isomorphisms. A non-example is the *Jordan quiver* consisting of one vertex and one self-loop.

THM 6.5 Gabriel

A connected quiver Q is of finite type, if and only if the underlying unoriented graph Γ is a Dynkin diagram.

Moreover, isomorphism classes of indecomposable representations of Q are in one-to-one correspondence with the positive roots associated to Γ, by $V \longleftrightarrow \dim V$.

Proof. For the only if part we refer to the homework (it requires a bit algebraic geometry). We now prove the if part.

If Q is a quiver, call $i \in I$ a **sink** or **source** if all edges connected to i points towards / away from i. If i is a sink or a source, let $\overline{Q_i}$ be the quiver obtained from Q by reversing all arrows connected to i.

We define the **reflection functor**, as follows. If i is a sink, define $F_i^+ : \mathrm{Rep}(Q) \to \mathrm{Rep}\left(\overline{Q_i}\right)$ by

$$\left(F_i^+(V)\right)_j = \begin{cases} V_j & j \neq i \\ \ker\left(\bigoplus_{k \to i} V_k \to V_i\right) & j = i \end{cases} \tag{73}$$

and for an edge $j \xrightarrow{h} k$, if $k \neq i$, x_h stays the same, and if $k = i$, then $j \to i$ becomes $j \leftarrow i$ and the map is the composition

$$\ker\left(\bigoplus_{l \to i} V_l \to V_i\right) \hookrightarrow \bigoplus_{l \to i} V_l \twoheadrightarrow V_j. \tag{74}$$

Similarly, if i is a source, define $F_i^- : \mathrm{Rep}(Q) \to \mathrm{Rep}\left(\overline{Q_i}\right)$ by

$$\left(F_i^-(V)\right)_j = \begin{cases} V_j & j \neq i \\ \operatorname{coker}\left(V_i \to \bigoplus_{i \to k} V_k\right) & j = i \end{cases} \tag{75}$$

and for an edge $j \xrightarrow{h} k$, if $j \neq i$, x_h stays the same, and if $j = i$, then $i \to k$ becomes $i \leftarrow k$ and the map is the composition

$$V_k \hookrightarrow \bigoplus_{i \to l} V_l \twoheadrightarrow \operatorname{coker}\left(V_i \to \bigoplus_{i \to l} V_l\right) \tag{76}$$

PROP 6.6

Suppose V is an indecomposable representation of Q.

- If i is a sink, then either $V \cong S(i)$, or $\bigoplus_{l \to i} V_l \to V_i$ is surjective.
- If i is a source, then either $V \cong S(i)$, or $V_i \to \bigoplus_{i \to l} V_l$ is injective.

COR 6.7 Let V be an indecomposable representation of Q. If i is a sink or a source, then $F_i^+(V)$ or $F_i^-(V)$ is either zero or indecomposable. If it is zero, then V is $S(i)$. If it is indecomposable, $\dim F_i^{\pm}(V) = s_i \cdot \dim V$.

Suppose Q is a Dynkin diagram. Fix a labelling by $1, 2, \cdots, n$ of Q wuch that $i < j$ if one can reach j from i. This labelling has the property that if n is a sink of Q, then $n-1$ is a sink of $\overline{Q}_n$. Therefore, we can consider a sequence of representations

$$V^{(0)} = V, V^{(1)} = F_n^+\left(V^{(0)}\right), V^{(2)} = F_{n-1}^+\left(V^{(1)}\right), \cdots, V^{(n)} = F_1^+\left(V^{(n-1)}\right). \tag{77}$$

Notice, however, that $V^{(n)}$ is again a representation of Q because every arrow is reversed twice. We have $\dim V^{(n)} = c \cdot \dim V$, where c is the Coxeter element. Therefore we can repeat the above process, replacing $V^{(0)}$ by $V^{(n)}$. This produces an infinite list of representations

$$\begin{aligned} &V^{(0)}, V^{(1)}, \cdots, V^{(n-1)}; \\ &V^{(n)}, V^{(n+1)}, \cdots, V^{(2n-1)}; \\ &\cdots \end{aligned} \tag{78}$$

This list of indecomposable representations must have a zero somewhere, since otherwise $\dim V^{(Mn)} = c^M \cdot \dim V$ will have a strictly negative component for some M. So there must be some m such that $V^{(m)} = S(p)$ (Lemma 6.3); take the smallest such m. Suppose $F_{i_1}^+ \cdots F_{i_m}^+(V) = S(p)$, then $V = F_{i_m}^- \cdots F_{i_1}^-(S(p))$, so $\dim V = s_{i_m} \cdots s_{i_1}\left(\alpha_p\right) \in R^+$. We have then produced a map from the indecomposable representations of Q to the positive roots, mapping V to $\dim V \in R^+$. This map is injective since it only depends on the initial dimension vector. To prove surjectivity suppose $\alpha \in R^+$. Analogously we apply simple reflections to α:

$$\begin{aligned} &\alpha, s_n\alpha, s_{n-1}s_n\alpha, \cdots, s_2 \cdots s_n\alpha; \\ &c\alpha, s_n c\alpha, s_{n-1}s_n c\alpha, \cdots, s_2 \cdots s_n c\alpha; \\ &\cdots \end{aligned} \tag{79}$$

This list of roots must have a negative root somewhere, since the first column contains an element with a negative component. Therefore, the last term to the first negative root in this list must be a simple root, since simple reflections s_i permute positive roots except for α_i itself. Suppose $s_{i_1} \cdots s_{i_l}\alpha = \alpha_p$. Let $V = F_{i_l}^- \cdots F_{i_1}^-(S(p))$, then $\dim V = \alpha$ and V is an indecomposable representation. □