Differential Geometry II

Riemannian Geometry

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Section 1 Preliminary Materials

For preliminary materials about differential geometry, we refer to the Differential Geometry I course notes.

1.1 Tensors and Tensor Fields

For a differential 1-form ω, we may regard it as a function accepting one vector field as the input: $\Gamma(TM) \to C^\infty(M)$, $X \mapsto \omega(X)$. Similarly, a vector field X is a function accepting one 1-form as the input: $\omega \mapsto \omega(X)$. They are both C^∞-**linear**: meaning that, for any $f \in C^\infty(M)$, $\omega(fX) = f\omega(X)$. They are examples of *tensor fields*.

Given a vector space V, a (k,l)**-tensor** is a multilinear map $T : (V^*)^{\otimes k} \otimes V^{\otimes l} \to \mathbb{R}$. k is called the **contravariant index**, the number of 1-form inputs, or the number of vector fields in the tensor; l is called the **covariant inex**, the number of vector field inputs, or the number of 1-forms in the tensor. If we let $V = T_pM$ and $V^* = T_pM$, we denote $T_p^{(k,l)}M$ the space of all (k,l)-tensors on T_pM. $T^{(k,l)}M$ is then a smooth manifold, the **tensor bundle**, and a smooth section is called a **tensor field**.

In local coordinates, a (k,l)-tensor field can be written in the form

$$T = T^{i_1\cdots i_k}_{j_1\cdots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_l}. \tag{1}$$

We're using Einstein summation convention here.

1.2 Lie Derivative

Lie derivatives are directional derivatives along a vector field.

Let X be a vector field. For a smooth function f, the Lie derivative is defined by $\mathcal{L}_X f = Xf$. **Lie derivatives** of tensors are defined inductively through the Leibniz rule:

$$\begin{aligned}
&(\mathcal{L}_X T)(\omega_1, \cdots, \omega_k; X_1, \cdots, X_l) \\
=\ &\mathcal{L}_X(T(\omega_1, \cdots, \omega_k; X_1, \cdots, X_l)) \\
&- \sum_{i=1}^{k} T(\omega_1, \cdots, \mathcal{L}_X \omega_i, \cdots, \omega_k; X_1, \cdots, X_l) \\
&- \sum_{j=1}^{l} T\big(\omega_1, \cdots, \omega_k; X_1, \cdots, \mathcal{L}_X X_j, \cdots, X_l\big).
\end{aligned} \tag{2}$$

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Section 2 Riemannian Manifolds

2.1 Riemannian Metric

DEF 2.1 A **Riemannian metric** g on M is a $(0,2)$-tensor field on M, such that for every $p \in M$, g_p is a symmetric positive definite bilinear form (i.e., an inner product) on T_pM.

A smooth manifold with a Riemannian metric is called a **Riemannian manifold**.

In local coordinates, g can be written as $g = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$; since g is symmetric we omit the $\otimes$ symbol. g transforms under different coordinates as

$$g_{ij} = g_{\alpha\beta} \cdot \frac{\partial y^\alpha}{\partial x^i} \cdot \frac{\partial y^\beta}{\partial x^j}. \tag{3}$$

EX 2.2

1. $\mathbb{R}^n$ with Euclidean metric: $\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + \cdots + (\mathrm{d}x^n)^2$.

   When $n = 2$, we can also use polar coordinates: $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$.

2. $\mathbb{S}^n$ with round metric, the metric restricted from the Euclidean metric on $\mathbb{R}^{n+1}$. For $n = 2$, it reads

$$\mathrm{d}s^2 = \frac{1}{1 - x^2 - y^2}\left(\left(1 - y^2\right)\mathrm{d}x^2 + \left(1 - x^2\right)\mathrm{d}y^2 + 2xy\,\mathrm{d}x\,\mathrm{d}y\right). \tag{4}$$

   Using spherical coordinates $(x, y, z) = (\sin r\cos\theta, \sin r\sin\theta, \cos r)$,

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \sin^2 r\,\mathrm{d}\theta^2. \tag{5}$$

   Using stereographic projection,

$$\mathrm{d}s^2 = \frac{4}{1 + \|x\|^2}\,\mathrm{d}s^2_{\text{Euclidean}}. \tag{6}$$

3. Hyperbolic space. The *Poincaré ball model*:

$$(\mathbb{D}^n, g),\ g = \frac{4}{\left(1 - \|x\|^2\right)^2}\,\mathrm{d}s^2_{\text{Euclidean}}; \tag{7}$$

   the *upper half space model*:

$$(\mathbb{R}^n_+, g),\ ,g = \frac{1}{x^n}62\,\mathrm{d}s^2_{\text{Euclidean}}; \tag{8}$$

   and the *Beltrami-Klein model*:

$$\begin{aligned}(\mathbb{D}^n, g),\ g = &\frac{\left(\mathrm{d}x^1\right)^2 + \cdots + (\mathrm{d}x^n)^2}{1 - \|x\|^2} \\ &+ \frac{\left(x^1\,\mathrm{d}x^1 + \cdots + x^n\,\mathrm{d}x^n\right)^2}{\left(1 - \|x\|^2\right)^2}.\end{aligned} \tag{9}$$

A diffeomorphism $\Phi : (M_1, g_1) \to (M_2, g_2)$ is called **conformal**, if $\Phi^* g_2 = e^f g_1$ for some $f \in C^\infty(M_1)$.

2.2 Length, Angle and Volume

Suppose (M_1, g_1) and (M_2, g_2) are Riemannian manifolds. If there is a diffeomorphism $\Phi : M - 1 \to M_2$ such that $\Phi^*(g_2) = g_1$, call M_1 and M_2 **isometric**, and Φ an **isometry**. Equivalently, $g_1(X, Y) = g_2(\Phi_* X, \Phi_* Y)$.

Using the metric we can define the length of a curve. Let $\gamma : (a, b) \to M$ be a smooth (or C^1) curve, define its **length**

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| \, \mathrm{d}t, \tag{10}$$

where $|\dot{\gamma}(t)| = \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}$. For $p, q \in M$, we can then define the **distance** between p and q, by

$$d(p,q) = \inf\{L(\gamma) : \gamma \text{ is a curve between } p \text{ and } q\}. \tag{11}$$

If M is orientable, under an oriented atlas, we can also define a **volume form**

$$\mathrm{d}v_g = \sqrt{\det\left(g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right)} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{12}$$

Given two smooth curves γ_1, γ_2 on M such that they intersect at $\gamma_1(t_1) = \gamma_2(t_2) = p$, we define the **angle** θ between them at p as

$$\cos\theta = \frac{g_p(\gamma'(t_1), \gamma'(t_2))}{\|\gamma'(t_1)\| \cdot \|\gamma'(t_2)\|}. \tag{13}$$

A diffeomorphism $\Phi : (M_1, g_1) \to (M_2, g_2)$ is called **conformal** if it preserves all angles. Equivalently, Φ is conformal if there is a smooth function $f \in C^\infty(M_1)$ such that $\Phi^*(g_2) = \mathrm{e}^f g_1$.

2.3 Riemannian Metric on Products

Suppose (M, g) and (N, h) are two Riemannian manifolds. We give the product manifold $M \times N$ a metric, the **product metric** $g \oplus h$, by

$$(g \oplus h)_{(p,q)}((v_1, w_1), (v_2, w_2)) = g_p(v_1, v_2) + h_q(w_1, w_2). \tag{14}$$

In local coordinates,

$$g \oplus h = \begin{pmatrix} g_{ij} & \\ & h_{kl} \end{pmatrix}. \tag{15}$$

The product metric can be generalized to **warped product metrics**, denoted $g \oplus_f h$, where $f \in C^\infty(M)$, defined as

$$\left(g \oplus_f h\right)_{(p,q)}((v_1, w_1), (v_2, w_2)) = g_p(v_1, v_2) + f(p) \cdot h_q(w_1, w_2). \tag{16}$$

The round metric on $\mathbb{S}^2$ in spherical coordinates form is an example of the warped product of two $\mathbb{R}$ with Euclidean metric.

2.4 Musical Isomorphisms

The Riemannian metric g provides a symmetric positive definite bilinear map $\Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$. This can also be viewed as a linear isomorphism $\Gamma(TM) \to \Gamma(T^*M)$, given by $v \mapsto g(v, -)$. In other words to every vector $v \in T_pM$, there is a covector $G(v)$ such that $G(v)(w) = g(v, w)$; this $G(v)$ is said to be obtained from v by **lowering the index**, and is denoted $v^\flat$. Similarly, for a covector α there is a corresponding vector $G^{-1}(v)$, obtained by **raising the index** it is denoted $\alpha^\sharp$.

A remark on the names "*lowering / raising* the indices". Normally, for a tensor, covariant indices are written as subscripts, and contravariant indices are written as superscripts. The name "*covariant*" and "*contravariant*" comes from the way they transform under a coordinate change: covariant tensors like 1-forms transforms by the Jacobian of the transformation

$$\mathrm{d}y^\alpha = \frac{\partial y^\alpha}{\partial x^i}\,\mathrm{d}x^i, \tag{17}$$

and contravariant tensors like vector fields transforms by the *inverse*

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial}{\partial x^i}. \tag{18}$$

Therefore, turning a (contravariant) vector into a covector makes it covariant, hence *lowering* its index from a superscript into a subscript; and turning a covector into a vector *raises* its index.

By raising the index, we can define a metric $\tilde{g}$ on $\Gamma(T^*M) \times \Gamma(T^*M)$, by $\tilde{g}(\alpha, \beta) = g\left(\alpha^\sharp, \beta^\sharp\right)$. In local coordinates,

$$\tilde{g}(\alpha, \beta) = g^{ij}\alpha_i\beta_j, \tag{19}$$

where $\alpha = \alpha_i\,\mathrm{d}x^i$, $\beta = \beta_j\,\mathrm{d}x^j$ and g^{ij} is the inverse of g_{ij}, $g^{ik}g_{kj} = \delta^i_j = \begin{cases} 1 \text{ if } i{=}j \\ 0 \text{ otherwise} \end{cases}$.

Similarly, we can define g on any (r, s)-tensor field:

$$g(T, S) = T^{i_1\cdots i_r}_{j_1\cdots j_s} S^{k_1\cdots k_r}_{l_1\cdots l_s} \cdot g_{i_1k_1}\cdots g_{i_rk_r} \cdot g^{j_1l_1}\cdots g^{j_sl_s}. \tag{20}$$

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Section 3
Parallel Transport

3.1 Affine Connections

Affine connections ∇ are a class of "*directional derivatives*". As all derivatives, we begin the definition with

$$\nabla_X f = Xf. \tag{21}$$

On vector fields, they are characterized by the following conditions:

DEF 3.1

1. $\nabla_X Y$ is $\mathbb{R}$-*linear* in Y and C^∞-*linear* in X. (Notice the difference!)
2. *Leibniz rule*: $\nabla_X(YZ) = Y(\nabla_X Z) + (\nabla_X Y)Z$.

If $(x^1, \cdots, x^n)$ is a local coordinates, $V = V^i \frac{\partial}{\partial x^i}$, $W = W^i \frac{\partial}{\partial x^i}$, then

$$\nabla_V V^i \nabla_{\frac{\partial}{\partial x^i}} \left(W^j \frac{\partial}{\partial x^j} \right) = V^i \left(\frac{\partial W^j}{\partial x^i} \frac{\partial}{\partial x^j} + W^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right). \tag{22}$$

Suppose that $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$; the Γ_{ij}^k are called **Christoffel symbols**. Moreover denote

$$\nabla_i W^j = \frac{\partial W^j}{\partial x^i} + W^k \Gamma_{ik}^j, \tag{23}$$

which is the usual Euclidean derivative plus a correction term. This is called the **tensor component**[1]. Then $\nabla_V W$ can be written locally as $V^i (\nabla_i W^j) \frac{\partial}{\partial x^j}$.

If ∇_1 and ∇_2 are two affine connections, then $\nabla_1 - \nabla_2$ is C^∞-linear in *both* variables, so it's a tensor field. This means that we can translate a connection with a tensor field to modify it.

The connection can also be defined on 1-forms using the Leibniz rule:

[1]the *tensor* in the name is just a *name*; it is not a tensor

$$(\nabla_X \alpha)(Y) = \nabla_x a(Y) - \alpha(\nabla_X Y). \tag{24}$$

This definition is also $\mathbb{R}$-linear in α, C^∞-linear in X, and satisfies the Leibniz rule. The tensor component is now

$$\nabla_i \alpha_j = \frac{\partial \alpha_j}{\partial x^i} - \alpha_k \Gamma^k_{ij}. \tag{25}$$

Generally, for (r,s)-tensor fields,

$$\nabla_X T = X^p \left(\nabla_p T^{i_1 \cdots i_r}_{j_1 \cdots j_s}\right) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}, \tag{26}$$

where the tensor component is

$$\begin{aligned} &\nabla_p T^{i_1 \cdots i_r}_{j_1 \cdots j_s} \\ &= \frac{\partial}{\partial x^p}\left(T^{i_1 \cdots i_r}_{j_1 \cdots j_s}\right) + \sum_k \Gamma^{i_k}_{p\beta} T^{i_1 \cdots \beta \cdots i_r}_{j_1 \cdots j_s} - \sum_l \Gamma^{\alpha}_{pj_l} T^{i_1 \cdots i_r}_{j_1 \cdots \alpha \cdots j_s}. \end{aligned} \tag{27}$$

The link of affine connections to Riemannian geometry is the following theorem:

THM 3.2 Suppose (M,g) is a Riemannian manifold. Then there is a *unique* affine connection ∇ on M, called the **Levi-Civita connection**, satisfying two more conditions:

1. *metric compatibility*: $\nabla_Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.[2]
2. *torsion-free*: $\nabla_X Y - \nabla_Y X = [X,Y]$.[3]

It can be expressed using Lie brackets and the inner product $\langle X,Y\rangle = g(X,Y)$ as

$$\begin{aligned} 2\langle Y, \nabla_Z X\rangle = Z\langle X,Y\rangle + X\langle Y,Z\rangle - Y\langle X,Z\rangle - \\ \langle X,[Z,Y]\rangle - \langle Z,[X,Y]\rangle - \langle Y,[X,Z]\rangle. \end{aligned} \tag{28}$$

In Christoffel symbols,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right). \tag{29}$$

3.2 Parallel Transport and Holonomy

Throughout this section, let ∇ be the Levi-Civita connection.

[2] Normally, according to Leibniz rule, there should be a third term on the right hand side, the term $(\nabla_Z g)(X,Y)$. For this reason, metric compatibility is also written as $\nabla g = 0$.

[3] Their difference, $\nabla_X Y - \nabla_Y X - [X,Y]$, is called the *torsion*, $T(X,Y)$.

DEF 3.3 Suppose $\gamma : (a, b) \to M$ is a curve and $V(t) \in T_{\gamma(t)}M$ is a vector field on γ. We say that V is **parallel** along γ if $\nabla_{\gamma'} V = 0$.

Generally, we say that a tensor T is **parallel** if for any vector field X, $\nabla_X T = 0$. In this case we write $\nabla T = 0$.

In local coordinates, V is parallel to γ if and only if $\frac{\mathrm{d}V^k}{\mathrm{d}t} + \Gamma^k_{ij}\gamma'^i(t)V^j = 0$, where Γ^k_{ij} can be calculated using Equation 29. This is a first-order linear ODE, therefore given an initial vector v at $\gamma(a)$, we can find a unique vector field V on γ such that $V(a) = v$. In this case, we say that $V(b)$ is obtained from $V(a)$ by **parallel transporting** along γ. Parallel transport along γ is denoted P_γ.

Suppose γ is a curve from p to q. Parallel transport is then a map from T_pM to T_qM. Moreover, it is a *linear isometry*, meaning that it is a linear map preserving the metric.

In particular, we can construct an orthonormal basis $\{E_1, \cdots, E_n\}$ of T_pM and parallel transport it along γ; it will remain orthonormal at any point. We thus obtain an orthonormal *frame* along the curve, $\{E_1(t), \cdots, E_n(t)\}$. If V is a vector field along the curve, then using this orthonormal frame, $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V^i}{\mathrm{d}t}E_i$.

Suppose α is a loop at p. Then P_α is a linear isometry from T_pM to T_pM itself. The set of parallel transport $\{P_\alpha : \alpha \text{ is a loop at } p\}$ then form a subgroup of $\mathrm{O}(n)$ under composition; this is called the **holonomy group** $\mathrm{Hol}_p(M, g)$. It is a Lie subgroup of $\mathrm{O}(n)$.

If we only consider contractible loops, then $\mathrm{Hol}^0_p(M, g) = \{P_\alpha : \alpha \text{ is contractible}\}$ is a connected component of $\mathrm{Hol}_p(M, g)$, and $\mathrm{Hol}_p/\mathrm{Hol}^0_p$ is at most countable.

The holonomy of product is the product of holonomy:

$$\mathrm{Hol}_{(p,q)}(M \times N, g \oplus h) = \mathrm{Hol}_p(M, g) \times \mathrm{Hol}_q(N, h). \tag{30}$$

Holonomy groups at different points are related by conjugation: if α is a curve from p to q, then $\mathrm{Hol}_p(M, g) = P_\alpha^{-1} \cdot \mathrm{Hol}_q(M, g) \cdot P_\alpha$.

A theorem by Ambrose-Singer says that the Lie algebra of Hol^0_p is generated by curvature forms.

Hol^0_p has a natural action on T_pM. Decompose T_pM into irreducible invariant subspaces $V_1 \oplus \cdots \oplus V_k$. Along parallel transport, the above decomposition preserves. Therefore, along a curve γ, $T_\gamma M$ can be decomposed into distributions $D_1 \oplus \cdots \oplus D_k$.

THM 3.4 The above decomposition is a local splitting around p: for some open neighbourhood of U, U is isometric to $(U_1 \times \cdots \times U_k, g_1 \oplus \cdots \oplus g_k)$.

If M is simply connected, the the above splitting can be made global, decomposing M into a product.

If a Riemannian manifold cannot be decomposed into a product, it is called *irreducible*.

THM 3.5 If M is simply connected and irreducible, then $\mathrm{Hol}_p = \mathrm{Hol}_p^0$ either acts on $\mathbb{S}^{n-1} \subseteq T_pM$ transitively, or M is a symmetric space of rank at least 2. In the first case we can make the following table:

$\dim M = n$	Hol_p	Properties
n	$\mathrm{SO}(n)$	orientable
$2n$	$\mathrm{U}(n)$	Kähler
$2n$	$\mathrm{SU}(n)$	Calabi-Yau (Ricci-flat Kähler)
$4n$	$\mathrm{Sp}(1)\mathrm{Sp}(n)$	Quaternion Kähler-Einstein
16	$\mathrm{Spin}(9)$	Symmetric Einstein
8	$\mathrm{Spin}(7)$	Ricci-flat
7	G_2	Ricci-flat
$\ldots$	$\ldots$	$\ldots$

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Section 4
Geodesics

DEF 4.1 A C^2-curve γ is called a **geodesic** if $\nabla_{\dot\gamma}\dot\gamma = 0$.

Locally, $(x_i(t))$ is a geodesic if and only if

$$\underbrace{\ddot{x}^i(t)}_{\substack{\text{Euclidean}\\\text{acceleration}\\\text{field}}} + \underbrace{\Gamma^i_{jk}\dot{x}^j(t)\dot{x}^k(t)}_{\substack{\text{non-Euclidean}\\\text{correction}}} = 0 \tag{31}$$

for all i. This is a (quasi-linear) ODE, so it has a unique solution under two initial conditions: $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$, and this unique solution is smooth.

If every maximal geodesic is defined on $\mathbb{R}$, or if every geodesic can be indefinitely extended, call this manifold **geodesic complete**.

4.1 Exponential map

Suppose $\gamma : [0,1] \to M$ is a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. We define the **exponential map** $\exp_p : T_pM \to M$ as $\exp_p(v) = \gamma(1)$.

If M is not geodesically complete, the exponential map may not be defined everywhere on T_pM. The domain of $\exp_p$ is always a star-shaped region (because $\exp_p(\lambda v) = \gamma(\lambda)$ for $\lambda < 1$). We often restrict it to a small ball around $0 \in T_pM$.

PROP 4.2 For any $p \in M$, there is some $\varepsilon > 0$, such that $\exp_p : B_\varepsilon(0) \subseteq T_pM \to M$ is a diffeomorphism onto its image. Moreover, $\left(\exp_p\right)_{*,0} = \mathrm{id}$.

In this case we call $\exp_p(B_{\varepsilon(0)})$ a **geodesic ball** and its boundary $\exp_p(S_\varepsilon(0))$ a **geodesic sphere**. $V \subseteq M$ is called a **normal neighbourhood** if there is some $U \in T_pM$ such that $\exp_p : U \to V$ is a diffeomorphism.

$\exp_p$ induces two sets of local coordinates around p. The first is the **normal coordinate**: it's the image of the orthonormal Cartesian coordinates on T_pM on a normal neighbourhood. Under normal coordinates, $\Gamma^k_{ij}(p) = 0$ and $g\left(e_i, e_j\right) = \delta_{ij}$ at p. (Notice: it's only valid *at* p, and not *around* p!)

The second is the **geodesic polar coordinates**. It is the image of polar coordinates on T_pM on a normal neighbourhood. Therefore the coordinates consists of radial geodesics from p and geodesic spheres around p. Marvelously, orthogonality of polar coordinates are preserved.

LEM 4.3 If U is a normal neighbourhood of p, then every geodesic from p is orthogonal to the geodesic spheres centered at p.

In particular, $\exp_p$ is an isometry *along the radial direction*: for any $v \in T_pM$ and $x \in T_pM \cong T_v\left(T_pM\right)$,

$$g\left(\left(\exp_p\right)_{*,v} x, \left(\exp_p\right)_{*,v} v\right) = g(x, v). \tag{32}$$

4.2 Hopf-Rinow Theorem

THM 4.4 A Riemannian manifold is geodesic complete,

- if and only if it is complete as a metric space,
- and if and only if exp is globally defined at some point (and hence every point).

As a corollary, all compact manifolds are complete.

4.3 Length Minimizers: First Variation of Arclength

Suppose $\gamma : [a, b] \to M$ is a (piecewise) smooth curve. A **variation family** Φ of γ is map $\Phi : [a, b] \times (-\varepsilon, \varepsilon) \to M$ such tht

1. $\Phi(-, 0) = \gamma$;
2. Φ is (piecewise in the first parameter) smooth.

We asy that Φ is **proper** if $\Phi(a, -) = \gamma(a)$ and $\Phi(b, -) = \gamma(b)$.

THM 4.5 If $\gamma : [a, b] \to M$ is a smooth curve and $|\dot\gamma(t)| = 1$, $\Phi(t, s)$ is a variational family of γ, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} L(\Phi(-, s)) = \langle v_0, T_0\rangle\Big|_a^b - \int_a^b \left\langle v_0, \nabla_{T_0} T_0\right\rangle \mathrm{d}t, \tag{33}$$

where $v_0 = \left.\Phi_*\left(\frac{\partial}{\partial s}\right)\right|_{s=0}$ and $T_0 = \left.\Phi_*\left(\frac{\partial}{\partial t}\right)\right|_{s=0}$.

As a corollary, if Φ is a proper variation and v_0 is chosen arbitrarily, then γ is arclength-minimizing only if γ is a geodesic.

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Section 5 Curvature

5.1 Hessian and Laplacian

If the Levi-Civita connection corresponds to the first order derivative, then the Hessian corresponds to second order derivative.

Recall that $\nabla_X f = X(f)$ can be treated as a covector field: $\nabla f : X \mapsto \nabla_X f$. By lowering the index, we get the **gradient vector field**

$$\operatorname{grad} f = (\nabla f)^\flat = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} =: \nabla^j f \cdot \frac{\partial}{\partial x^j}. \tag{34}$$

$\nabla\nabla f$ is then a $(2,0)$-tensor field: $\nabla\nabla f : (X, Y) \mapsto (\nabla_X(\nabla f))(Y)$. By some calculation, $\nabla\nabla f = X(Y(f)) - (\nabla_X Y)(f)$. This is called the **Hessian**. Locally:

$$(\nabla\nabla f)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}. \tag{35}$$

Therefore, the Hessian is symmetric in X and Y. Conventionally we write $\nabla\nabla f = \nabla_i \nabla_j f \cdot \mathrm{d}x^i\,\mathrm{d}x^j$, where

$$\nabla_i \nabla_j f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}. \tag{36}$$

Similarly, $\nabla Y : X \mapsto \nabla_X Y$ is a $(1,1)$-tensor field. Locally,

$$\nabla Y = \left(\frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk} Y^k\right)\mathrm{d}x^j. \tag{37}$$

The **Laplacian** of a function f is defined as $\Delta f = \operatorname{tr}\nabla(\operatorname{grad} f)$. Locally,

$$
\begin{aligned}
\Delta f &= \operatorname{tr} \nabla\left(\nabla^i f \cdot \frac{\partial}{\partial x^i}\right) \\
&= \operatorname{tr}\left(\nabla_j \nabla^i f \cdot \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j\right) \\
&= \nabla_i \nabla^i f = \nabla_j\left(g^{jk} \nabla_k f\right) \\
&= g^{jk} \nabla_j \nabla_k f.
\end{aligned} \tag{38}
$$

The last step is by metric compatibility. Actually, we have the following formula:

$$
\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x^j}\right) = \operatorname{div} \operatorname{grad} f. \tag{39}
$$

5.2 Riemannian Curvature

For vector fields we can do similar things. Let X, Y and Z be vector fields. Then

$$
\begin{aligned}
\nabla_X \nabla_Y Z &= X^i \nabla_i\left(Y^j \nabla_j Z^k \frac{\partial}{\partial x^k}\right) \\
&= X^i \nabla_i\left(Y^j \nabla_j Z^k\right) \frac{\partial}{\partial x^k} \\
&= X^i\left(\nabla_i Y^j \nabla_j Z^k + Y^j \nabla_i \nabla_j Z^k\right) \frac{\partial}{\partial x^k}
\end{aligned} \tag{40}
$$

Therefore,

$$
\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = X^i Y^j\left(\nabla_i \nabla_j Z^k - \nabla_j \nabla_i Z^k\right) \frac{\partial}{\partial x^k} + \nabla_{[X,Y]} Z. \tag{41}
$$

Remarkably, the first term, which is equal to $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, is a tensor. This is the **Riemannian curvature tensor** $R(X,Y)Z$. This is a $(1,3)$-tensor field, and it can be written as $R = R_{ij\ k}^{\ \ l}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial l} \otimes x^k$. The i, j, k indices corresponds to the X, Y, Z input, and the l index corresponds to the field $R(X,Y)Z$.

Expanding the defining equation into Christoffel symbols, we get

$$
R_{ij\ l}^{\ \ k} = \left(\partial_i \Gamma_{jl}^k + \Gamma_{ip}^k \Gamma_{jl}^p\right) - \left(\partial_j \Gamma_{il}^k + \Gamma_{jq}^k \Gamma_{il}^q\right). \tag{42}
$$

By lowering the index l we get a $(0,4)$-tensor field

$$
\begin{aligned}
R_m : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) &\to C^\infty(M), \\
X, \quad Y, \quad Z, \quad W &\mapsto \langle R(X,Y)W, Z\rangle.
\end{aligned} \tag{43}
$$

In local coordinates, $R_{ijkl} = g_{kp} R_{ij\ l}^{\ \ p}$.

PROP 5.1

These are symmetric properties of R and R_m.

In global form:

- $R(X,Y)Z = -R(Y,X)Z$;
- $R_m(X,Y,Z,W) = -R_m(Y,X,Z,W) = -R_m(X,Y,W,Z) = R_m(Z,W,X,Y)$;
- (*first Bianchi identity*) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$;
- (*first Bianchi identity*) $R_m(X,Y,W,Z) + R_m(Y,Z,W,X) + R_m(Z,X,W,Y) = 0$; the same holds if we fix any input and cycle the rest.

In local form:

- $R_{ij\ k}^{\ \ l} = -R_{ji\ k}^{\ \ l}$;
- $R_{ij\ k}^{\ \ l} + R_{jk\ i}^{\ \ l} + R_{ki\ j}^{\ \ l} = 0$;
- $R_{ijkl} + R_{jlki} + R_{likj} = 0$; the same holds if we fix any index and cycle the rest.
- $R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij}$.

$R(X,Y) : Z \mapsto R(X,Y)Z$ is called the **curvature operator**. Locally, $R\left(\partial_i, \partial_j\right)Z = \left(Z^l R_{ij\ l}^{\ \ k}\right)\partial_l$, so

$$\nabla_i \nabla_j Z^k - \nabla_j \nabla_i Z^k = R_{ij\ l}^{\ \ k} Z^l. \tag{44}$$

Extending $R(X,Y)$ to $\Omega^1(M)$,

$$\begin{aligned} R(X,Y)\alpha &= \nabla_X \nabla_Y \alpha - \nabla_Y \nabla_X \alpha - \nabla_{[X,Y]}\alpha \\ \implies \nabla_i \nabla_j \alpha_k - \nabla_j \nabla_i \alpha_k &= -R_{ij\ k}^{\ \ l}\alpha_l = R_{ijk}^{\ \ \ l}\alpha_l. \end{aligned} \tag{45}$$

For functions, $R(X,Y)f = 0$, because the Hessian is symmetric.

By anti-symmetric properties of R_m, it is actually a map $\wedge^2 TM \otimes \wedge^2 TM \to C^\infty(M)$, sending $(X \wedge Y, Z \wedge W)$ to $R_m(X,Y,Z,W)$. Moreover, we can define an operator $R : \wedge^2 TM \to \wedge^2 TM$, such that $\langle R(X \wedge Y), Z \wedge W\rangle = R_m(X \wedge Y, Z \wedge W)$. This R is self-adjoint, and is called the **curvature operator**.

THM 5.2
Hamilton

Any compact 4-dimensional Riemannian manifold with a positive curvature operator is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$.

Any compact 4-dimensional Riemannian manifold with a non-negative curvature operator is diffeomorphic to $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{R}^4$ or their quotient by a finite group.

5.3 Sectional Curvature, Ricci Curvature, Scalar Curvature

DEF 5.3
sectional curvature

Let $p \in M$ and σ be a 2-dimensional tangent plane spanned by $X_p, Y_p \in T_pM$. Define the **sectional curvature** at (p,σ):

$$K_p(\sigma) = \frac{R_m\left(X_p, Y_p, X_p, Y_p\right)}{\left\|X_p \wedge Y_p\right\|^2}. \tag{46}$$

Here $\|u \wedge v\|^2 = g(u,u)g(v,v) - g(u,v)^2$ is the infinitesimal area spanned by u and v. This definition is independent of the basis X_p, Y_p chosen.

In dimension 2, there can be only one tangent plane at every point, and the sectional curvature is equal to the Gaussian curvature: $K = \frac{R_{1212}}{\|\partial_1 \times \partial_2\|^2}$.

The sectional curvature can be defined for any $(0,4)$-tensor satisfying the symmetry properties in Proposition 5.1. Conversely, if for two such $(0,4)$-tensors R_m and $\tilde{R}_m$, we get the same sectional curvature at p for all σ ($K_p(\sigma) = \tilde{K}_p(\sigma)$), then $R_m(p) = \tilde{R}_m(p)$.

If for any p and any σ, $K_p(\sigma)$ is constant, we call M a manifold of **constant sectional curvature**. If M has constant curvature $c \in \mathbb{R}$, then

$$R_m(X,Y,Z,W) = c \cdot (g(X,Z)g(Y,W) - g(X,W)g(Y,Z)). \tag{47}$$

If M is a simply connected manifold of constant curvature, then M must be isometric to a sphere, the Euclidean space, or a hyperbolic space.

DEF 5.4 Ricci curvature, scalar curvature

Let $p \in M$ and $X \in T_pM$ with norm 1. Take an orthonormal basis $\{X = e_1, e_2, \cdots, e_n\}$.

Define the **Ricci curvature** $\mathrm{Ric}_p(X) = \sum_{i=2}^n R_m(X, e_i, X, e_i)$. The **Ricci tensor** is a $(0,2)$-tensor $\mathrm{Ric}(X,Y) = \sum_{i=1}^n R_m(X, e_i, Y, e_i)$. Locally, we can write $\mathrm{Ric} = R_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$ ($\otimes$ is omitted since this is symmetric), where $R_{ik} = g^{jl}R_{ijkl}$.

Define the **scalar curvature** by $S = g^{ik}R_{ik} = g^{ik}g^{jl}R_{ijkl}$. Equivalently, $S = \sum_{i=1}^n \mathrm{Ric}(e_i, e_i)$.

PROP 5.5 second Bianchi identity

$$\nabla_X R(Y,Z,U,V) + \nabla_Y R(Z,X,U,V) + \nabla_Z R(X,Y,U,V) = 0. \tag{48}$$

Locally,

$$\nabla_i R_{jkpq} + \nabla_j R_{kipq} + \nabla_k R_{ijpq} = 0. \tag{49}$$

Taking contraction by g^{ip} and g^{jq}, we get

$$\nabla^i R_{ik} = \frac{1}{2}\nabla_k S. \tag{50}$$

The geometric meaning of this identity is encoded in the **Hilbert-Einstein functional** $\mathrm{HE}(g) = \int S_g\, \mathrm{dvol}_g$. By the second Bianchi identity, this is diffeomorphism invariant.

LEM 5.6 Schur

Let (M,g) be a Riemannian manifold with dimension at least 3, and satisfies one of the following conditions:

1. $K_p(\sigma)$ doesn't depend on σ (and only on p): $K_p(\sigma) = f(p)$.
2. $\mathrm{Ric}(p) = f(p) \cdot g_p$ for some smooth function f; i.e., for all $v \in T_pM$, $\mathrm{Ric}_p(v) = f(p) \cdot |v|^2$.

Then f must be a constant.

5.4 The Laplacian Operator

We introduce some analysis-flavoured results on manifolds. The key is the following expansion of the Laplacian:

THM 5.7 Bochner's formula

For any smooth function $f \in C^\infty(M)$,

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla\nabla f|^2 + \langle \nabla(\Delta f), \nabla f \rangle + \mathrm{Ric}(\nabla f, \nabla f). \tag{51}$$

Similar formulas also exist for Laplacians of (harmonic) 1-forms, etc.

The typical use of this formula is in certain special cases, for example, when Ric is positive / negative definite, when f is a harmonic function (so that $\nabla(\Delta f) = \lambda \nabla f$), etc. The following theorem is useful in working with harmonic functions:

THM 5.8 Hopf-Calabi maximum principle

Let (M, g) be a connected Riemannian manifold, f is a smooth function on M such that $\Delta f \geqslant 0$. Such a function is called **subharmonic**.

Then f attains no maximum in M, unless f is constant.

Therefore, for example, to prove a function on a compact manifold is constant, one only need to prove that it's subharmonic, and we can use Bochner's formula to estimate the Laplacian.

Some examples of statements that can be proved using these results:

PROP 5.9

Suppose M is a compact manifold.

- If $\mathrm{Ric}\, M < 0$ (i.e. negative definite), then M admits no Killing field (fields such that $\mathcal{L}_X g = 0$).
- If $\mathrm{Ric}\, M > 0$, then M admits no harmonic 1-form ($\Delta\omega = 0$). (Therefore, by Hodge theory, $H^1(M; \mathbb{R}) = 0$.)

6

Section 6

Jacobi Fields and Second Variation

DEF 6.1
Jacobi fields

Suppose γ is a geodesic, and $J(t)$ is a vector field on γ. Then $J(t)$ is called a **Jacobi field** if it satisfies the following equation:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J(t) = R(\dot\gamma(t), J(t))\dot\gamma(t). \tag{52}$$

Jacobi fields arise in variational families of geodesics. Formally, if $F(s,t) = \gamma_s(t) : (-\varepsilon, \varepsilon) \times [0,a] \to M$ is a variational family of geodesics, i.e. γ_s is a geodesic for all s, then the variational vector field $J(t) = \frac{\partial F}{\partial s}\Big|_{s=0}$ is a Jacobi field. Conversely, if $J(t)$ is a Jacobi field along a geodesic γ, then there is a variational family of geodesics $F(s,t) = \gamma_s(t)$ such that $J(t) = \frac{\partial F}{\partial s}\Big|_{s=0}$. The construction can be visualized by the following diagram.

Choosing a parallel frame $\{E_i\}$ along γ and set $J(t) = f^i(t)E_i(t)$, then the equation may be written as a system of second order linear ODEs:

$$\ddot{f^j} = f^i \cdot R_m\big(\dot\gamma, E_i, E_j, \dot\gamma\big). \tag{53}$$

Let γ be a geodesic. $J(t) = \dot\gamma(t)$ is a Jacobi field, representing parallel translation of γ along itself, $\gamma_s(t) = \gamma(t+s)$. $J(t) = t\dot\gamma(t)$ is also a Jacobi field, representing a linear reparametrization of gamma, $\gamma_s(t) = \gamma((1+s)t)$. By linearity of the Jacobi field equation, $(a+bt)\dot\gamma(t)$ are all Jacobi fields. In fact, they are the only parallel Jacobi fields.

Therefore, all Jacobi fields can be decomposed as $J^\perp(t) + (a+bt)\dot\gamma(t)$, where $J^\perp(t)$ is orthogonal to $\dot\gamma(t)$; such a Jacobi field is called a **normal Jacobi field**.

Suppose $J(0) = 0$. Then Taylor expanding $|J(t)|^2$ about t gives

$$|J(t)|^2 = t^2|w|^2 - \frac{1}{3}R(v,w,v,w)t^4 + o\big(t^4\big), \tag{54}$$

where $v = \gamma'(0)$ and $w = J'(0)$. Therefore, in negative sectional curvature, $J(t)$ is "longer" than in the Euclidean case, and in positive sectional curvature, $J(t)$ is "shorter" than in the Euclidean case (think hyperbolic planes and spheres). Similarly, in normal coordinates,

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l + o\left(|x|^2\right). \tag{55}$$

6.1 Conjugate Loci

Suppose $\gamma : [0, a] \to M$ is a geodesic and $t_0 \in [0, a]$. If there is a Jacobi field along γ vanishing at both $\gamma(0)$ and $\gamma(t_0)$, call $\gamma(t_0)$ a **conjugate point** of, or **conjugate** to, $\gamma(0)$. The number of linearly independent such Jacobi fields is called the **multiplicity** of this conjugate point.

If $\gamma'(0) = v$, then $\gamma(t_0)$ is conjugate to $\gamma(0)$ if and only if $t_0 v$ is a critical point of $\left(\exp_p\right)_*$, i.e. there is some w such that $\left(\exp_p\right)_{*, t_0 v}(w) = 0$. If $\gamma(t_0)$ is not conjugate to $\gamma(0)$, then there is a unique Jacobi field from 0 to t_0, given arbitrary boundary conditions $J(0) = v$, $J(a) = w$.

The **conjugate locus** of p is defined as

$$C(p) = \{q \in M : q \text{ is the first conjugate point of } p \text{ along some geodesic.}\} \tag{56}$$

6.2 Second Variation of Arclength and the Index Form

THM 6.2 second variation of arclength

Suppose γ is a geodesic, γ_s is a variational family of γ with variational vector field V. Denote $V^\perp$ the normal component (with respect to γ). Then

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} L(\gamma_s) = \int_a^b \left(\left|\nabla_{\dot\gamma} V^\perp\right|^2 - R(\dot\gamma, V^\perp, \dot\gamma, V^\perp) \right) \mathrm{d}t + \left.\langle \nabla_V V, \dot\gamma \rangle\right|_a^b. \tag{57}$$

If we fix the endpoints, the boundary term vanishes, and we are left with

$$\int_a^b \left(\left|\nabla_{\dot\gamma} V^\perp\right|^2 - R(\dot\gamma, V^\perp, \dot\gamma, V^\perp) \right) \mathrm{d}t. \tag{58}$$

We can make this into a bilinear form in two vector fields, by defining the **index form**

$$I(V, W) = \int_a^b \left(\left\langle \nabla_{\dot\gamma} V, \nabla_{\dot\gamma} W \right\rangle - R(\dot\gamma, V, \dot\gamma, W) \right) \mathrm{d}t. \tag{59}$$

The index form is a symmetric bilinear form; its null space is the space of all Jacobi fields along γ vanishing at the endpoints.

For a variational vector field V, $I(V,V) \geqslant I(V^\perp, V^\perp) = \frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s)$. Therefore, a normalized geodesic γ is arclength-minimizing *only if* the index form along γ is positive semi-definite.

LEM 6.3 index lemma

Suppose $\gamma : [0,t] \to M$ is a normalized geodesic from p to q containing no conjugate point to p. Let W be a vector field along γ with $W(0) = 0$, and let J be the unique Jacobi field with $J(0) = W(0) = 0$ and $J(t) = W(t)$.

Then $I(J,J) \leqslant I(W,J)$, equality holds if and only if $W = J$.

THM 6.4

Geodesics are no longer minimal beyond the conjugate locus.

Suppose $\gamma : [0,l] \to M$ is a normalized geodesic from p. If $\gamma(t_0)$ is conjugate to p, then $\gamma|_{[0,t_1]}$ is not length minimizing for any $t_1 > t_0$.

6.3 Effects of Curvature on Global Topology

6.3.1 Manifolds of Positive Curvature

Jacobi fields and conjugate loci are closely related to the global topological shape of the manifold. This can be seen from the "model space" $\mathbb{S}^n$: it has positive curvature, and any point is conjugate to its antipodal point. The following theorem says that this is a general phenomenon for manifold with positive curvature:

THM 6.5 Bonnet-Myers

Let (M,g) be an n-dimensional complete Riemannian manifold. Assume either that the sectional curvature K has positive lower bound c, or that the Ricci curvature Ric has positive lower bound $(n-1)c$.

Then every geodesic of length larger than $\frac{\pi}{\sqrt{c}}$ has a conjugate point, and hence not length-minimizing. As a corollary, M must be compact with diameter no larger than $\frac{\pi}{\sqrt{c}}$.

Moreover, $\pi_1(M)$ must be finite.

Proof. Suppose γ is a normalized geodesic with length $L \geqslant \frac{\pi}{\sqrt{c}}$. Let $\{e_i\}$ be a parallel frame along γ. Our goal is to construct a vector field W such that $I(W,W) < 0$.

To do this we make a reference to the "model space" $\mathbb{S}^n$, where Jacobi fields are given by $J_i(t) = \sin\left(\frac{\pi}{L}t\right)e_i(t)$. Let

$$W_i(t) = \frac{L}{\pi}\sin\left(\frac{\pi}{L}t\right)e_i(t). \tag{60}$$

Then $W_i(0) = W_i(L) = 0$, and

$$
\begin{aligned}
I(W_i, W_i) &= \cancel{\langle W_i, \nabla_{\dot\gamma} W_i \rangle\big|_0^L} - \int_0^L \langle W_i, \nabla_{\dot\gamma}\nabla_{\dot\gamma} W_i - R(\dot\gamma, W)\dot\gamma \rangle \,\mathrm{d}t \\
&= \int_0^L \left\langle \frac{L}{\pi} \sin\left(\frac{\pi}{L}t\right) e_i(t), \frac{\pi}{L} \sin\left(\frac{\pi}{L}t\right) e_i(t) + \frac{L}{\pi} \sin\left(\frac{\pi}{L}t\right) R(\dot\gamma, e_i)\dot\gamma \right\rangle \mathrm{d}t \\
&= \int_0^L \left(\sin^2\left(\frac{\pi}{L}t\right) - \left(\frac{\pi}{L}\right)^2 \sin^2\left(\frac{\pi}{L}t\right) R(\dot\gamma, e_i, \dot\gamma, e_i) \right) \mathrm{d}t \\
&= \int_0^L \sin^2\left(\frac{\pi}{L}t\right) \left(1 - \left(\frac{\pi}{L}\right)^2 R(\dot\gamma, e_i, \dot\gamma, e_i) \right) \mathrm{d}t.
\end{aligned}
\tag{61}
$$

In the case that the sectional curvature is larger than or equal to c, the term $1 - \left(\frac{\pi}{L}\right)^2 R(\dot\gamma, e_i, \dot\gamma, e_i)$ is negative. In the case that Ricci curvature is larger than or equal to $(n-1)c$, we add the $I(W_i, W_i)$ up for all i, and notice that the terms $1 - \left(\frac{\pi}{L}\right)^2 R(\dot\gamma, e_i, \dot\gamma, e_i)$ sum up to $(n-1) - \left(\frac{\pi}{L}\right)^2 \operatorname{Ric}(\dot\gamma, \dot\gamma)$, which is again negative. Therefore, at least one of the $I(W_i, W_i)$ is negative. Therefore γ is not minimizing.

Since M is complete with diameter no larger than $\frac{\pi}{\sqrt{c}}$, $M \subseteq \exp_p\left(B_0\left(\frac{\pi}{\sqrt{c}}\right)\right)$ for any point c. Therefore, M is compact. Moreover its universal covering $\widetilde{M}$ also satisfy the conditions of this theorem, so $\widetilde{M}$ is also compact. Therefore, the universal covering of M is a finite covering. This proves the finiteness of $\pi_1(M)$. □

THM 6.6 Weinstein

Suppose M is an n-dimensional compact Riemannian manifold with positive sectional curvature. If either

- n is even and $f : M \to M$ is an orientation-preserving isometry, or
- n is odd and $f : M \to M$ is an orientation-reversing isometry,

then f necessarily has a fixed point.

Together with Theorem 6.5, we deduce useful topological information about manifolds with positive sectional curvature:

COR 6.7 Synge

if M is an n-dimensional compact Riemannian manifold with positive sectional curvature, then

- if n is even and M is oriented, then M is simply connected; and
- if n is odd, then M is orientable.

For non-compact manifolds, we have an even stronger result:

THM 6.8 Cheeger-Gromov

If M is non-compact with positive sectional curvature, then any isometry of M share a fixed point, called the *soul* of M; and M is always diffeomorphic to $\mathbb{R}^n$.

Proof of Theorem 6.6. The function $d(p, f(p))$, as a continuous function on a compact space, has a minimum point q. If f has no fixed point, then $d(q, f(q)) >$

0. Take a normaliezd geodesic γ from q to $f(q)$. We wish to find a variational family γ_s such that $\frac{\mathrm{d}^2}{\mathrm{d}s^2}L(\gamma_s) < 0$, arriving at a contradiction.

Consider

$$A : T_qM \xrightarrow{f} T_{f(q)}M \xrightarrow{\text{parallel transport along } \gamma} T_qM. \tag{62}$$

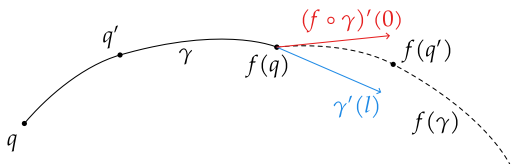
This is a linear isometry on T_qM. We claim that $\gamma'(0)$ is an eigenvector of this linear isometry. Restricting A onto the orthogonal complement of $\gamma'(0)$, it is in $\mathrm{O}^+(n-1)$ if n is even, or $\mathrm{O}^-(n-1)$ if n is odd. Either case it must have another eigenvector $e_1(0)$. Parallel transport $e_1(0)$ to form a vector field $e_1(t)$ along γ. Let the variational field be $h(s,t) = \exp_{\gamma(t)} e_1(t) \cdot s$. Then

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} L(\gamma_s) = \int_0^l \Big(\cancel{\langle \nabla_{\dot\gamma} e_1, \nabla_{\dot\gamma} e_1 \rangle} - R(\dot\gamma, e_1, \dot\gamma, e_1)\Big)\,\mathrm{d}t + \cancel{\langle \nabla_{e_1(t)} e_1(t), \dot\gamma \rangle \Big|_0^l} < 0 \tag{63}$$

We now prove the claim that $\gamma'(0)$ is an eigenvalue of A, or equivalently $(f \circ \gamma)'(0) = \gamma'(l)$. This is equivalent as proving that the angle between γ and $f(\gamma)$ is exactly π.

Take q' on γ between $\gamma(0)$ and $\gamma(l)$. Then

$$\begin{aligned} d(q', f(q')) &\leqslant d(q', f(q)) + d(f(q), f(q')) && \text{since } f \text{ is an isometry} \\ &= d(q', f(q)) + d(q, q') && \text{since } q' \text{ lies on the geodesic } \gamma \\ &= d(q, f(q)). \end{aligned} \tag{64}$$



By minimality equality holds, so γ and $f(\gamma)$ as a whole form a geodesic. This proves the claim. □

6.3.2 Manifolds of Non-Positive Curvature

Recall from earlier that $J(t) = t - \frac{K}{6}t^3 + o\big(t^3\big)$. This shows that, locally, larger sectional curvature yields shorter Jacobi fields. The following theorem extends this globally:

THM 6.9 Rauch comparison theorem

Suppose M_i $(i = 1, 2)$ are Riemannian manifolds, $\gamma_i(t)$ are normalized geodesics on M_i with $t \in (0, a)$ with no conjugate points, J_i are Jacobi fields along γ_i with $J_1(0) = J_2(0) = 0$, $\langle J_1'(0), \gamma_1'(0) \rangle = \langle J_2'(0), \gamma_2'(0) \rangle$ and $|J_1'(0)| = |J_2'(0)|$, and for all t and vector fields X_i along γ_i, $K(X_1, \gamma_1') \leqslant K(X_2, \gamma_2')$.

Then $|J_1| \geqslant |J_2|$.

Actually, $\frac{|J_1|}{|J_2|}$ is non-decreasing in t. Therefore, if at any point equality holds, the sectional curvatures must also be equal.

COR 6.10 With Bonnet-Myers theorem, if $L \leqslant K \leqslant H$, then $\frac{\pi}{\sqrt{H}} \leqslant \operatorname{diam} M \leqslant \frac{\pi}{\sqrt{L}}$.

COR 6.11 Suppose M_1, M_2 are Riemannian manifolds, $K_1 \geqslant K_2$, $p_i \in M_i$, $r > 0$ such that $\exp_{p_i}\big|_{B_r(0)}$ are diffeomorphisms, $i : T_{p_1}M_1 \to T_{p_2}M_2$ a linear isometry, $C_1 : [0, a] \to \exp_p(B_0(r))$ a smooth curve and $C_2 = \exp_{p_2} \circ i \circ \exp_{p_1}^{-1}$.

Then $L(C_1) \leqslant L(C_2)$.

Therefore, comparison of sectional curvatures yields comparisons of lengths. Analogously, since the Ricci curvature is the average of sectional curvatures, comparison of Ricci curvatures yields comparisons of areas.

We have discussed much about manifolds with non-negative sectional curvature. The following theorem analyzes the case of non-*positive* sectional curvature, using the comparison theorem:

THM 6.12 Cartan-Hadamard

Suppose M is complete with non-positive sectional curvature. Then for any p, $\exp_p : T_pM \to M$ is a covering map.

If M is moreover simply connected, then $\exp_p$ is a diffeomorphism.

As a corollary, if M is a complete manifold with non-positive sectional curvature, then the universal covering of M is diffeomorphic to $\mathbb{R}^n$. In particular M is a $K(\pi_1(M), 1)$. Such a manifold is called a **Cartan-Hadamard manifold**.

6.3.3 Manifolds of Constant Sectional Curvature

How much information can we recover from the curvature of a manifold? More precisely, if there is a diffeomorphism $f : M \to \widetilde{M}$ between manifolds of the same dimension and f preserves curvature, then is f an isometry? The answer is no: in dimension $n \leqslant 3$, there are M and f such that f is curvature-preserving, M is not constand curvature yet f is not an isometry. However, in dimension $n \geqslant 4$, Kulkarni proved that if M is analytic, then either M has constant curvature or f is an isometry. In general, for f to be isometry, M needs to have "enough anisotropy"; if $n \geqslant 4$, M is smooth and the anisotropic points are dense, then f is an isometry. If $n \geqslant 3$, then f is a conformal map on the closure of anisotropic points. If $n \geqslant 3$ and M is nowhere constantly curved, then f must be an isometry. However, the answer to the local version is affirmative: curvature-preserving *infinitesimal* diffeomorphisms can be integrated into a local isometry.

THM 6.13 Cartan

Let $p \in M$, $\tilde{p} \in \widetilde{M}$, $i : T_pM \to T_{\tilde{p}}\widetilde{M}$ a linear isometry, $B_p(r)$ a normal neighbourhood of p, where r small enough to ensure that $\exp_p$ is a diffeomorphismin the neighbourhood.

Let $f = \exp_{\tilde{p}} \circ i \circ \left(\exp_p\right)^{-1}$. Let $\gamma(t)$ be a normalized geodesic inside $B_p(r)$ such that $\gamma(0) = p$. Let $\tilde{\gamma} = f \circ \gamma$, and let P_γ be the parallel transport along γ. Let $i_t = P_{\tilde{\gamma}} \circ i \circ \left(P_\gamma\right)^{-1} : T_{\gamma(t)}M \to T_{\tilde{\gamma}(t)}\widetilde{M}$ be the "infinitesimal diffeomorphisms".

If for all $q = \gamma(t) \in B_p(M)$ and all $X, Y, Z, W \in T_qM$ we have $R(X, Y, Z, W) = \tilde{R}(i_tX, i_tY, i_tZ, i_tW)$, then f is a local isometry and $(f_*)_p = \mathrm{id}$.

If this is the case, then $f_{*,\gamma(t)} = i_t$.

COR 6.14

1. If $\exp_p$ and $\exp_{\tilde{p}}$ are diffeomorphisms from $\mathbb{R}^n$ to M or $\widetilde{M}$ and $K_M = K_{\widetilde{M}}$, then f is a *global* isometry.
2. If M and $\widetilde{M}$ has equal constant sectional curvature c, then there is a local isometry $f : M \to \widetilde{M}$ around p such that f takes any given orthonormal frame at p to any given orthonormal frame at $\tilde{p}$.

   In particular, the isometries of M itself are transitive on points.

A theorem by Cartan-Ambrose-Hicks says that if M and $\widetilde{M}$ are moreover simply connected, then the local isometry in Theorem 6.13 can be extended to a global one.

We can use this result to classify manifolds of constant sectional curvature. A complete Riemannian manifold with constant sectional curvature c is called a **space form**. In this case, $R_{ijkl} = c\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$.

THM 6.15 If M is a simply-connected space form and $c = +1 \,/\, -1 \,/\, 0$, then M is isometric to $\mathbb{S}^n \,/\, \mathbb{H}^n \,/\, \mathbb{R}^n$.

Proof. The case $c = -1$ and $c = 0$ is covered by Theorem 6.12 and Theorem 6.13. We now assume $c = 1$, and we try to construct an isometry $\mathbb{S}^n \to M$. The idea is to create this map by two charts.

Let $p \in \mathbb{S}^n$, $\tilde{p} \in M$ and $i : T_p\mathbb{S}^n \to T_{\tilde{p}}M$ be a linear isometry. Let q be the antipode of p. Then by Theorem 6.13, there is a local isometry $f_1 : \mathbb{S}^n \setminus \{q\} \to M$, $f_1 = \exp_{\tilde{p}} \circ i \circ \left(\exp_p\right)^{-1}$ taking p to $\tilde{p}$.

Take another point $p' \in M$ distinct from p and q, let $\tilde{p}' = f_1(p)$, and $i' : T_{p'}\mathbb{S}^n \to T_{\tilde{p}'}M$ be a linear isometry. Let q' be the antipode of p', then similarly there is a local isometry $f_2 : \mathbb{S}^n \setminus \{q'\} \to M$, $f_2 = \exp_{\tilde{p}'} \circ i' \circ \left(\exp_{p'}\right)^{-1}$, also taking p to $\tilde{p}$. Moreover, $(f_1)_{*,p} = (f_2)_{*,p}$. Therefore, f_1 and f_2 agrees wherever they are both defined. They together gives a local isometry $f : \mathbb{S}^n \to M$. Since both $\mathbb{S}^n$ and M are simply connected, f is an isometry. □

If M is any space form, then M must be the quotient of $\widetilde{M} = \mathbb{S}^n \,/\, \mathbb{H}^n \,/\, \mathbb{R}^n$ by a totally disconnected group action $\Gamma \subset \mathrm{Isom}\left(\widetilde{M}\right)$. (*Totally disconnected* means that for any $q \in \widetilde{M}$, there is some neighbourhood U of q such that for all $g \in \Gamma$, $g(U) \cap U = \emptyset$.) This Γ is the group of deck transformations of the covering map $\widetilde{M} \to M$. For example, if M is $2n$-dimensional with constant sectional curvature 1, then M must be isometric to either $\mathbb{S}^{2n}$ or $\mathbb{RP}^{2n}$.

6.4 Comparison Theorems

We have proved earlier a result on comparison of Jacobi fields, the Rauch comparison theorem.

Let $p \in M$ and define the distance function $\rho(x) = d(x,p)$. Recall that the index form is the second variation of arclength, so we might expect that the second derivative of ρ is related to the index form. Actually this is the case.

For any $q \in M$, let $\gamma : [0,l] \to M$ be a minimizing geodesic from p to q, so $\rho(\gamma(t)) = |\dot\gamma(0)| \cdot t$. Suppose $X \in T_qM$ orthogonal to $\dot\gamma(l)$, and we extend X into a Jacobi field J along γ orthogonal to $\dot\gamma$. J is thus a variational field of γ. We compute the Hessian of ρ at J:

$$\begin{aligned}
\operatorname{Hess}\rho(J,J) &= JJ\rho - (\nabla_J J)\rho \\
&= J\langle J, \nabla\rho\rangle - \langle \nabla_J J, \nabla\rho\rangle \\
&= \cancel{\langle \nabla_J J, \nabla\rho\rangle} + \langle J, \nabla_J(\nabla\rho)\rangle - \cancel{\langle \nabla_J J, \nabla\rho\rangle} \\
&= \langle J, \nabla_J \nabla\rho\rangle \\
&= \langle J, \nabla_J \dot\gamma\rangle \\
&= \langle J, \nabla_{\dot\gamma} J\rangle.
\end{aligned} \tag{65}$$

The second to last equality is because γ is a minimizing geodesic, implying $\nabla\rho = \dot\gamma$ along γ. Therefore,

$$\begin{aligned}
\operatorname{Hess}\rho(X,X) &= \int_0^l \frac{\mathrm{d}}{\mathrm{d}t}\langle J, \nabla_{\dot\gamma} J\rangle \,\mathrm{d}t \\
&= \int_0^l \left(|\nabla_{\dot\gamma} J|^2 - \langle J, \nabla_{\dot\gamma}\nabla_{\dot\gamma} J\rangle\right) \mathrm{d}t \\
&= \int_0^l \left(|\nabla_{\dot\gamma} J|^2 + R(\dot\gamma, J, J, \dot\gamma)\right) \mathrm{d}t \\
&= I(J,J),
\end{aligned} \tag{66}$$

where I is the index form.

Define the **cut point** of a geodesic γ as follows. Take the supremum t_0 of all t such that γ is minimizing on $(0,t)$. If such supremum is finite, call $\gamma(t_0)$ the *cut point* of γ. The **cut locus** of a point p, $\operatorname{Cut}(p)$, is the set of cut points of normalized geodesics coming out from p. If M is compact, then the cut locus of a point is never empty. Notice that there are compact examples and non-compact examples of manifolds that do not have conjugate points, but do have cut loci.

THM 6.16 Hessian comparison

Suppose (M_i, g_i) are complete, $K_1 > K_2$, $\gamma_i : [0,l] \to M_i$ are normalized minimizing geodesics, $X_i \in T_{\gamma_i(l)}M$, $|X_i| = 1$ and $X_i \perp \dot\gamma_i(l)$.

Then $\operatorname{Hess} \rho_1(X_1, X_2) \leqslant \operatorname{Hess} \rho_2(X_2, X_2)$.

The trace, or "average in all directions", of Hessian, is the Laplacian, and the average of sectional curvatures is the Ricci curvatures. We get comparison of Laplacians from Ricci curvatures:

THM 6.17 Laplacian comparison

Let M be complete, N a simply connected space form with sectional curvature k. Assume $\operatorname{Ric} M \geqslant (n-1)k$. Let $p \in M$, $q \in N$, $\rho_M(x) = d_M(x, p)$ and $\rho_N(y) = d_N(y, q)$.

Then $\Delta\rho_M \leqslant \Delta\rho_N$.

COR 6.18

If $\operatorname{Ric} M \geqslant -(n-1)k^2$, then $\Delta\rho \leqslant \frac{n-1}{\rho}(1 + k\rho)$.

If $\operatorname{Ric} M \geqslant 0$, then $\Delta\rho \leqslant \frac{n-1}{\rho}$.

Hessian comparison corresponds to length comparison, so analogously Ricci comparison corresponds to volume comparison.

THM 6.19 Bishop-Gromov comparison theorem

Suppose M is complete, N a simply connected space form with sectional curvature k, $\operatorname{Ric} M \geqslant (n-1)k$, $x \in M$, $\tilde{x} \in N$. Consider

$$\frac{\operatorname{Vol}(B_x(R))}{\operatorname{Vol}(B_{\tilde{x}}(R))}, \quad \frac{\operatorname{Area}(\partial B_x(R))}{\operatorname{Area}(\partial B_{\tilde{x}}(R))}. \tag{67}$$

These two functions in R are non-increasing in R. Since they approach to 1 as $R \to 0$, in particular,

$$\operatorname{Vol}(B_x(R)) \leqslant \operatorname{Vol}(B_{\tilde{x}}(R)), \quad \operatorname{Area}(\partial B_x(R)) \leqslant \operatorname{Area}(\partial B_{\tilde{x}}(R)). \tag{68}$$

7

Section 7
Riemannian Submanifolds

DEF 7.1 An immersion $i : M^n \looparrowright \widetilde{M}^{n+k}$ is called **isometric**, if $g = i^*\tilde{g}$. M is called a **Riemannian submanifold**, if i is an isometric embedding.

At any point $p \in M$, $T_p\widetilde{M}$ can be decomposed into T_pM and its orthogonal complement N_pM. N_pM is called the **normal space** of M at p. This gives a splitting of the tangent bundle, $0 \to TM \to T\widetilde{M} \to NM \to 0$, where NM is called the **normal bundle**.

Let $X, Y \in \Gamma(TM)$, $\tilde{X}$ and $\tilde{Y}$ are local extensions of i_*X and i_*Y. At $p \in M$, $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ can be decomposed into a tangential part and a normal part, $\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \left(\tilde{\nabla}_{\tilde{X}}\tilde{Y}\right)^\top + \left(\tilde{\nabla}_{\tilde{X}}\tilde{Y}\right)^\perp$. The tangential part is equal to the Levi-Civita connection on $(M, i^*\tilde{g})$, denoted $\nabla_X Y$.

The difference of two connections is tensorial. Therefore, $\left(\tilde{\nabla}_{\tilde{X}}\tilde{Y}\right)^\perp = \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \nabla_X Y$ is tensorial on M. This is called the **second fundamental form**, denoted $\mathrm{II}(X, Y)$.

Suppose $X, Y \in \Gamma(TM)$ and N a normal vector field.

PROP 7.2 Weingarten equation

$$\tilde{g}(\tilde{\nabla}_{\tilde{X}}N, Y) = -\tilde{g}(\mathrm{II}(X, Y), N). \tag{69}$$

Based on this equation, define the **shape operator** $S_N(X) = -\left(\tilde{\nabla}_{\tilde{X}}N\right)^\top$. Then S_N is a self-adjoint operator $\Gamma(TM) \to \Gamma(TM)$. Therefore, it can be diagonalized, and its eigenvalues are called the **principal curvatures**.

Let $\alpha : [0, l] \to M$ be a smooth curve in M. Assume $|\dot{\alpha}| = 1$. Pick a parallel orthonormal frame $\{\dot{\alpha}, e_2, \cdots, e_n\}$ along α. Since $\langle\dot{\alpha}, \dot{\alpha}\rangle = 1$, taking $\nabla_{\dot{\alpha}}$, we get $\langle\nabla_{\dot{\alpha}}\dot{\alpha}, \dot{\alpha}\rangle = 0$. Let $k(t) = |\nabla_{\dot{\alpha}}\dot{\alpha}|$, called the **geodesic curvature**. α is a geodesic if and only it its geodesic curvature is zero.

Extend $\dot{\alpha}$ locally, then $\tilde{\nabla}_{\tilde{\dot{\alpha}}}\tilde{\dot{\alpha}} = \nabla_{\dot{\alpha}}\dot{\alpha} + \mathrm{II}(\dot{\alpha},\dot{\alpha})$. So a geodesic in M may not be a geodesic in $\widetilde{M}$. If every geodesic of M is also a geodesic of $\widetilde{M}$, call M **totally geodesic** this is the case if and only if $\mathrm{II} \equiv 0$.

7.1 Curvature of Submanifolds

Suppose $X \in \Gamma(TM)$, $N \in \Gamma(NM)$. $\tilde{\nabla}_X N$ can be split into a tangential part and a normal part; the tangential part is the shape operator. The second term is a connection on NM, called the **normal connection**, denoted $\nabla^{\perp}$. Define the **normal curvature**, $R^{\perp}(X,Y)N = \nabla^{\perp}_X\nabla^{\perp}_Y N - \nabla^{\perp}_X\nabla^{\perp}_Y N - \nabla^{\perp}_{[X,Y]}N$.

The Riemann curvature tensor R on M can be seen as an operator $\tilde{R}(X,Y)$: $\binom{TM}{NM} \to \binom{TM}{NM}$. Therefore, it can be written as a block matrix, and each block gives an equation describing $\tilde{R}$:

$$\tilde{R}(X,Y) = \begin{pmatrix} \text{Gauss} & \text{Codazzi} \\ -\text{Codazzi} & \text{Ricci} \end{pmatrix}. \tag{70}$$

Gauss equations relate the Riemannian curvature tensor R on M to the *tangential part* of $\tilde{R}$:

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) - \tilde{g}(\mathrm{II}(X,Z),\mathrm{II}(Y,W)) + \tilde{g}(\mathrm{II}(X,W),\mathrm{II}(Y,Z)), \\ \tilde{R}(X,Y)Z &= R(X,Y)Z - S_{\mathrm{II}(Y,Z)}X + S_{\mathrm{II}(X,Z)}Y. \end{aligned} \tag{71}$$

Codazzi equations describes the *normal* part of $\tilde{R}$. Denote $B(X,Y,Z) = \langle \mathrm{II}(X,Y),N\rangle$:

$$\begin{aligned} \left(\tilde{R}(X,Y)Z\right)^{\perp} &= \left(\nabla^{\perp}_X \mathrm{II}\right)(Y,Z) - \left(\nabla^{\perp}_Y \mathrm{II}\right)(X,Z), \\ \left(\tilde{R}(X,Y)Z\right)^{\perp} &= \mathrm{II}(X,\nabla_Y Z) - \mathrm{II}(Y,\nabla_X Z) - \mathrm{II}([X,Y],Z) \\ &\quad + \left(\tilde{\nabla}_X \mathrm{II}(Y,Z) - \tilde{\nabla}_Y \mathrm{II}(X,Z)\right)^{\perp}, \\ \left\langle \tilde{R}(X,Y)Z,N \right\rangle &= \left(\tilde{\nabla}_X B\right)(Y,Z,N) - \left(\tilde{\nabla}_Y B\right)(X,Z,N). \end{aligned} \tag{72}$$

Ricci equations describes the *normal curvature* $R^{\perp}$:

$$\begin{aligned} \left\langle \tilde{R}(X,Y)N_1,N_2 \right\rangle - \left\langle R^{\perp}(X,Y)N_1,N_2 \right\rangle &= \left\langle \left[S_{N_2},S_{N_1}\right]X,Y \right\rangle, \\ \left(\tilde{R}(X,Y)N\right)^{\perp} &= R^{\perp}(X,Y)N + \mathrm{II}(S_N(X),Y) - \mathrm{II}(S_N(Y),X). \end{aligned} \tag{73}$$

If we take a local *orthonormal* frame $e_1,\cdots,e_n$ of TM and $E_{n+1},\cdots,E_{n+k}$ or NM, and denote i,j,k for indices 1 to n and α,β,γ for indices $n+1$ to $n+k$, then we have the local formulas

$$\mathrm{II}\left(e_i,e_j\right) = \sum_{\gamma} h^{\gamma}_{ij}E_{\gamma}, \quad S_{E_\alpha}(e_i) = \sum_j h^{\alpha}_{ij}e_j, \tag{74}$$

(lowering and raising of indices is not needed because the frame is orthonormal) and the Gauss equation, Codazzi equation and Ricci equation can be written as

$$\begin{aligned}\tilde{R}_{ijkl} &= R_{ijkl} - \sum_{\alpha} h^{\alpha}_{ik} h^{\alpha}_{jl} + \sum_{\beta} h^{\beta}_{il} h^{\beta}_{kj},\\ \tilde{R}_{ij\ k}^{\ \ \alpha} &= \tilde{\nabla}_i h^{\alpha}_{jk} - \tilde{\nabla}_j h^{\alpha}_{ik},\\ \tilde{R}_{ij\ \alpha}^{\ \ \beta} &= R_{ij\ \alpha}^{\ \ \beta} + \sum_k \left(h^{\alpha}_{ik} h^{\beta}_{kj} - h^{\alpha}_{jk} h^{\beta}_{ki} \right).\end{aligned} \tag{75}$$

7.2 Hypersurfaces in Euclidean Spaces

A **hypersurface** is a submanifold of codimension 1.

Suppose M^n is a hypersurface in $\mathbb{R}^{n+1}$, and denote the Euclidean derivative by D. Since there is only one normal direction, we fix a normal field N and we can write $\mathrm{II}(X, Y) = h(X, Y)N$ where h is a matrix smooth in x and y, and

$$\begin{aligned} & \mathrm{D}_X Y = \nabla_X Y + h(X, Y)N,\\ \text{(Weingarten)}\quad & h(X, Y) = \langle \mathrm{II}(X, Y), N \rangle = \langle S_N(X), Y \rangle,\\ \text{(Gauss)}\quad & R(X, Y, Z, W) = \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W) \rangle - \langle \mathrm{II}(X, W), \mathrm{II}(Y, Z) \rangle,\\ \text{(Codazzi)}\quad & (\mathrm{D}_X h)(Y, Z) = (\mathrm{D}_Y h)(X, Z).\end{aligned} \tag{76}$$

The Ricci equation is degenerate in this case.

Since S_N is self-adjoint, it is diagonalizable. Suppose e_i are normal eigenvectors, so they are automatically orthonormal and $S_N e_i = \lambda_i e_i$. e_i are called the **principal directions**, and λ_i are called the **principal curvatures**. Define the **Gauss curvature** by $K = \det S_N = \lambda_1 \cdots \lambda_n$, and the **mean curvature** by $H = \frac{1}{n} \operatorname{tr} S_N = \frac{1}{n}(\lambda_1 + \cdots + \lambda_n)$. They can be written equivalently as $K = \frac{\det \mathrm{II}}{\det \mathrm{I}}$ and $H = \frac{1}{n} \operatorname{tr}_{\mathrm{I}} \mathrm{II}$, where $\mathrm{I} = g$ is the first fundamental form.

If we take local coordinates x^i for M and $\boldsymbol{n}$ a unit normal, then we can write $\mathrm{II}\left(\partial_i, \partial_j\right) = h_{ij}\boldsymbol{n}$, $S_{\boldsymbol{n}}(\partial_i) = -\mathrm{D}_i \boldsymbol{n} = h_i^j \partial_j$ and

$$\begin{aligned} K = \frac{\det h_{ij}}{\det g_{ij}} = \det h_i^j, \quad H = \frac{1}{n} \operatorname{tr}_g h = \frac{1}{n} g^{ij} h_{ij},\\ R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \quad \mathrm{D}_i h_{jk} = \mathrm{D}_k h_{ik}.\end{aligned} \tag{77}$$

The coefficients h_{ij} can be computed by $h_{ij} = \left\langle \mathrm{D}_i \partial_j, \boldsymbol{n} \right\rangle = \left\langle \partial_j, -\mathrm{D}_i \boldsymbol{n} \right\rangle$.

If $H \equiv 0$, M is called a **minimal hypersurface** because it is the minimizer of the area functional.

Suppose M is closed and oriented. The **Gauss map** $G : M \to \mathbb{S}^n$ is defined as the unit normal at that point. We generally uses the inner normal. The differential of G is equal to the negative of the shape operator: $\mathrm{d}G_p(X) = \nabla_X \boldsymbol{n}_p = -S_{\boldsymbol{n}}(X)$.

Let $\rho(x) = d(x, M)$ for x in the bounded region of $\mathbb{R}^{n+1} \setminus M$. Then $\boldsymbol{n} = -\nabla\rho$ (recall that $\boldsymbol{n}$ is the inner normal) and therefore $\mathrm{d}G = \nabla\boldsymbol{n} = -\nabla^2\rho$, so $S_{\boldsymbol{n}} = \nabla^2\rho$, and $h(X, Y) = \operatorname{Hess}\rho(X, Y)$, $H = \Delta\rho$.

8

Section 8
Method of Moving Frames

Let M be a Riemannian manifold, U a local chart and take orthonormal frame (vector fields) $\{e_1, \cdots, e_n\}$ and let $\{\omega^1, \cdots, \omega^n\}$ be the *coframe* (such that $\omega^i e_j = \delta^i{}_j$).

Suppose $\nabla_X e_i = e_j \omega^j{}_i(X)$. The 1-forms $\omega^j{}_i$ are called the **connection** 1**-forms**. If ∇ is the Levi-Civita connection, then

$$\begin{aligned} 0 = \nabla_X \langle e_i, e_j \rangle &= \langle \nabla_X e_i, e_j \rangle + \langle e_i, \nabla_X e_j \rangle \\ &= \langle e_k \omega^k{}_i X, e_j \rangle + \langle e_i, e_k \omega^k{}_j X \rangle \\ &= \left(\omega^j{}_i + \omega^i{}_j \right) X, \end{aligned} \tag{78}$$

so the matrix $\omega = \left(\omega^j{}_i \right)$ of 1-forms is antisymmetric, or $\mathfrak{so}(n)$-valued.

Suppose $R(X,Y)e_j = e_i \Omega^i{}_j(X,Y)$. The 2-forms $\Omega^i{}_j$ are called the **curvature** 2 **-forms** and $\Omega^i{}_j = \frac{1}{2} R_{kl}{}^i{}_j \omega^k \wedge \omega^l$. By symmetries of R, $\Omega = \left(\Omega^i{}_j \right)$ is also anti-symmetric, or $\mathfrak{so}(n)$-valued.

THM 8.1 Cartan structure formulas

1. $\Omega = \omega \wedge \omega + \mathrm{d}\omega$.
2. $\mathrm{d}\omega^i = \omega^j \wedge \omega^i{}_j$.